Finite amplitude cellular convection

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Summary

When a layer of fluid is heated uniformly from below and cooled from above, a cellular regime of steady convection is set up at values of the Rayleigh number exceeding a critical value. A method is presented here to determine the form and amplitude of this convection. The non-linear equations describing the fields of motion and temperature are expanded in a sequence of inhomogeneous linear equations dependent upon the solutions of the linear stability problem. We find that there are an infinite number of steady-state finite amplitude solutions (having different horizontal plan-forms) which formally satisfy these equations. A criterion for 'relative stability' is deduced which selects as the realized solution that one which has the maximum mean-square temperature gradient. Particular conclusions are that for a large Prandtl number the amplitude of the convection is determined primarily by the distortion of the distribution of mean temperature and only secondarily by the self-distortion of the disturbance, and that when the Prandtl number is less than unity self-distortion plays the dominant role in amplitude determination. The initial heat transport due to convection depends linearly on the Rayleigh number; the heat transport at higher Rayleigh numbers departs only slightly from this linear dependence. Square horizontal plan-forms are preferred to hexagonal plan-forms in ordinary fluids with symmetric boundary conditions. The proposed finite amplitude method is applicable to any model of shear flow or convection with a soluble stability problem.

Introduction

This is a study of the non-linear advective processes which determine the form and amplitude of cellular convection. We shall pose our problem as a formal extension of the conventional linearized stability theory. One purpose is to advance a bit closer to the formidable problem of the onset of turbulence.

Linearized stability theory determines those conditions of a known steady field of flow which first permit the growth of an infinitesimal disturbance. The amplitude of the preferred disturbance or class of preferred disturbances is found from the theory to grow exponentially in time for values of the external parameters in excess of a critical value.

In actuality such disturbances do not grow exponentially without limit, but by advecting heat and momentum they alter their own form and the distribution of their energy sources to achieve a finite equilibrium amplitude. How does this amplitude depend on the external parameters? How much heat and momentum are advected? What distortion of the form of the disturbance occurs at various amplitudes? What determines the preferred state of motion when the stability problem is degenerate, i.e. when more than one solution is possible at the point of instability? Under what conditions does this finite disturbance itself become unstable? Is the new field a similar cellular disturbance or has it the form of an aperiodic 'turbulent' motion?

We have chosen to investigate the steady-state, finite-amplitude, vertical convection of heat for several reasons. It is perhaps the simplest manifestation of non-linear advection in fluids, both geometrically and in the equipment needed for controlled experiments. The stability problem has been exactly solved in terms of trigonometric and hyperbolic functions (Rayleigh 1916; Pellew & Southwell 1940). Some detailed experimental data on the heat transport due to the convection at and beyond the point of instability exist (Malkus 1954a) to compare with theoretical deductions.

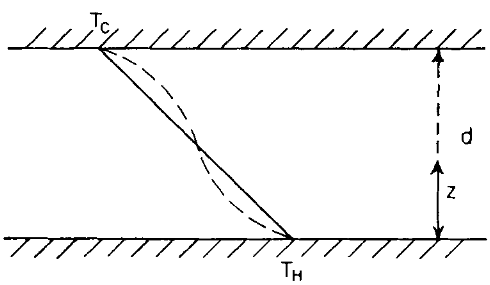

Figure 1. Geometry of the convection problem.

The physical situation to be studied is shown schematically in figure 1. The fluid is contained between two extensive horizontal conducting surfaces distance d apart. The upper surface is held at the temperature T_C and the lower surface at the higher temperature T_H. In the absence of motion the temperature distribution in the fluid is determined solely by the thermal conductivity and is indicated by the heavy line connecting T_H and T_C. In 1916 Lord Rayleigh deduced the criterion for marginal stability in such a fluid layer. This criterion involves what has come to be called the Rayleigh number

$$\lambda \equiv \frac{\alpha g}{\kappa \nu} \beta_m d^4$$

where α is the linear coefficient of expansion of the fluid, g is the magnitude of the gravitational acceleration, $\beta_m = (T_H - T_C)/d$ is the mean negative temperature gradient, κ is the thermometric conductivity and ν the kinematic viscosity of the fluid. When λ exceeds a critical value convection occurs as a regular cellular pattern. The theory does not distinguish between rectangular, triangular or hexagonal horizontal plan-forms for these cells. All could occur at the same value of λ with identical exponential growth rates. Another undetermined quantity in the linear theory is the sign of vertical motion in the centre of the hexagons.

The observations recorded in the earlier experimental paper (Malkus 1954 a) establish that for values of λ up to ten times the critical value λ_c a steady cellular convection exists in the fluid. The heat transport due to this convection shows a remarkably linear increase from zero at the critical λ to a value greater than that due to the conduction alone at $\lambda = 10\lambda_c$. This is the range of λ to be studied in this paper.

At $\lambda \doteqdot 10\lambda_c$ a new instability occurs in the fluid producing disordered aperiodic motions, quasi-cellular in appearance. This has been interpreted as the onset of some type of turbulence. However, the heat transport once again increases linearly with λ, but with a steeper slope than in the cellular region. Further discrete changes which steepen the slope of the heat transport curve appear at higher values of λ and appear to be associated with further instabilities. Each transition leads to a more intense and apparently more disordered field of motion. A preliminary statistical study of the convection at these high values of λ has been made in an earlier paper (Malkus 1954 b). Several results of the present study of cellular convection which relate to the aperiodic convection at large λ will be discussed in the conclusion.

The dashed curve connecting T_C and T_H in figure 1 is an example of the distribution of horizontally-averaged temperature when cellular convection occurs. The boundary conditions prevent convective heat transport at the boundaries. Hence the additional heat transport is permitted by increased mean temperature gradients at the boundary and decreased gradients in the mid-regions of the fluid. One task we now undertake is to determine the magnitude and shape of this distortion of the mean temperature field.

In the first section we describe the iterative method of solution for the finite amplitude fields of temperature and velocity. This is followed by a detailed treatment of the simplest case, that of two-dimensional convection (roll cell) for free surface boundary conditions. We were able to carry this analysis to 'sixth order' and investigate many of the 'distortions' of the initial convective disturbance. In § 3 we study the first finite amplitude effects of cellular convection with rectangular and hexagonal plan-forms (three-dimensional cells). Up to this point a multiply-infinite set of steady-state finite amplitude solutions has been generated, and we turn to the problem of their stability. The 'relative stability' argument of § 4 resolves the indeterminancy of the steady-state equations by providing a criterion which selects the realizable solutions. In § 5 we discuss the finite amplitude solutions when the fluid has rigid boundaries and in conclusion describe certain approximation techniques to extend the useful range of the formal analysis.

1. Basic equations and the method of solution

The equations which have been used by past authors to investigate convective instability are attributed to Boussinesq (1903, p. 173). They are based on the assumptions that the viscosity and thermal conductivity of the fluid can be treated as constants and that variations in the initial density field are important only in the buoyancy-force term in the equations of motion. Jeffreys (1930) has shown that the resulting disturbance equation is valid in compressible fluids if the temperature gradient minus the adiabatic lapse rate is used rather than the actual temperature gradient and if the total variations in density in the fluid are very small compared to the mean density. In liquids these equations for the local conservation of heat, momentum, and mass respectively are

$$(\partial/\partial t-\kappa\nabla^2)\mathbf{T}=-\mathbf{V}.\nabla\mathbf{T}, \tag{1.1}$$

$$(\partial/\partial t-\nu\nabla^2)\mathbf{V}=-\mathbf{V}.\nabla\mathbf{V}-\nabla\tilde{P}/\rho_0+\gamma(\mathbf{T}-\mathbf{T}_0)\mathbf{k}, \tag{1.2}$$

$$\nabla.\mathbf{V}=0, \tag{1.3}$$

where $\mathbf{V}$ is the vector velocity of the fluid, $\mathbf{T}$ is the actual temperature, $\mathbf{T}_0$ is a standard temperature and ρ_0 a standard density. $\tilde{P}=P-gz$ where P is the pressure, $\gamma=\alpha g$, and $\mathbf{k}$ is a unit vector in the z-direction. The non-linear advective terms $\mathbf{V}.\nabla\mathbf{T}$ and $\mathbf{V}.\nabla\mathbf{V}$ are to be retained in this analysis and are the principal objects of interest. However, in applying these equations to finite amplitude processes it must be borne in mind that the maximum density fluctuation from the mean must be small. Several interesting geophysical and astrophysical applications of this work require reappraisal of the assumptions and the inclusion of additional terms in the equations.

To clarify the role of the non-linear terms they will be divided into terms which are finite when averaged over a horizontal plane and which therefore modify the average field, and into terms of zero average.

Denoting an average over a horizontal plane by a bar above the quantity, the heat flux equation (1.1) becomes

$$\frac{\partial\overline{\mathbf{T}}}{\partial t}+\frac{\partial}{\partial z}(\kappa\beta)=-\frac{\partial}{\partial z}(\overline{WT}) \tag{1.4}$$

where $\beta=-\partial\overline{\mathbf{T}}/\partial z$, $T=\mathbf{T}-\overline{\mathbf{T}}$, and W is the vertical component of $\mathbf{V}$. Using (1.4), (1.1) may be rewritten as

$$(\partial/\partial t-\kappa\nabla^2)T=\beta W-h, \tag{1.5}$$

where $$h=\mathbf{V}.\nabla T-\partial(\overline{WT})/\partial z, \qquad \overline{h}=0.$$

From (1.2) and (1.5) two useful relations can be generated which describe the gross energetics of any flow. Multiplying (1.5) by $\gamma T/\beta_m$, (1.2) by $\mathbf{V}$, and averaging the resulting equations over the entire fluid, one obtains

$$\frac{1}{2}\frac{\gamma}{\beta_m}\frac{\partial}{\partial t}(\overline{T^2})_m=\gamma\left(\frac{\beta}{\beta_m}\overline{WT}\right)_m-\frac{\gamma\kappa}{\beta_m}(\overline{\nabla T.\nabla T})_m, \tag{1.6}$$

and

$$\frac{1}{2}\frac{\partial}{\partial t}(\overline{\mathbf{V}.\mathbf{V}})_m=\gamma(\overline{WT})_m-\nu(\overline{\nabla V_i.\nabla V_i})_m, \tag{1.7}$$

where the subscript m denotes the mean value over a vertical line joining the two horizontal boundaries, and V_i denotes the i-component of the velocity. The averages over the triple-product terms generated by the zero average non-linear terms and the average over the work term $\mathbf{V}.\nabla P$ must all vanish since they represent conservative advections within the fluid system (implying that $(\overline{A\mathbf{V}.\nabla A})_m = 0$ when $\nabla.\mathbf{V} = 0$ and A is any fluctuating scalar field).

Equation (1.7) is the 'power integral' of the motion. It states that the time rate of change of kinetic energy per unit mass is equal to the rate of release of potential energy by the convection minus the rate of dissipation of kinetic energy by the viscous stresses. Equation (1.6) has been written as a 'power integral' to parallel (1.7). However (1.6) is more correctly interpreted as either the balance equation for the mean square of fluctuations of the internal energy, or, equivalently, as an entropy balance equation. The particular value of these non-linear integral statements is in the determination of amplitude when the form of the motion is known or adequately approximated (e.g. Meksyn & Stuart 1951). We shall consider them later in this paper.

Since we will pivot our analysis about the linearized stability problem, it is convenient to eliminate all but one of the dependent variables appearing as linear terms in (1.2), (1.3) and (1.5). Therefore, we cross differentiate to remove the pressure and the linear terms involving the two horizontal velocity components U and V in (1.2) and (1.3). The resulting relation between W and T (plus non-linear terms) is

$$(\partial/\partial t - \nu\nabla^2)\nabla^2 W = \gamma\nabla_1^2 T + L(\mathbf{M}), \tag{1.8}$$

where

$$L(\mathbf{M}) \equiv \frac{\partial^2 M_x}{\partial x \partial z} + \frac{\partial^2 M_y}{\partial y \partial z} - \nabla_1^2 M_z,$$

$$M_x = \mathbf{V}.\nabla U, \qquad M_y = \mathbf{V}.\nabla V, \qquad M_z = \mathbf{V}.\nabla W,$$

and

$$\nabla_1^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2.$$

The two equations which relate the linear components U and V to W are

$$\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\left(\nabla_1^2 U + \frac{\partial^2 W}{\partial x \partial z}\right) = -\frac{\partial^2 M_x}{\partial y^2} + \frac{\partial^2 M_y}{\partial x \partial y}, \tag{1.9}$$

$$\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\left(\nabla_1^2 V + \frac{\partial^2 W}{\partial y \partial z}\right) = \frac{\partial^2 M_x}{\partial x \partial y} - \frac{\partial^2 M_y}{\partial x^2}. \tag{1.10}$$

The linear parts of (1.8), (1.9) and (1.10) are just those found in the analysis of convective instability. Cross differentiating (1.8) and (1.5) produces the usual sixth-order equation in the one 'linear' variable W:

$$\left(\frac{\partial}{\partial t} - \kappa\nabla^2\right)\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\nabla^2 W - \gamma\beta\nabla_1^2 W = -\gamma\nabla_1^2 h + \left(\frac{\partial}{\partial t} - \kappa\nabla^2\right)L(\mathbf{M}). \tag{1.11}$$

All terms on the right of (1.11) are non-linear terms with zero average. On the left are linear terms except for the important horizontal-average temperature gradient which depends upon $\overline{WT}$. To establish this latter relation we first note that the time-independent integral of (1.4) for fixed boundary temperatures is

$$\kappa\beta + \overline{WT} = H = \kappa\beta_m + (\overline{WT})_m, \tag{1.12}$$

where H is the constant heat flux between the horizontal surfaces due to both conduction ($\kappa\beta$) and convection ($\overline{WT}$). Then from the general integral of (1.4) one may write

$$\frac{\beta}{\beta_m} = 1 + \frac{1}{\kappa\beta_m}\left\{(\overline{WT})_m - \overline{WT} + G\left(\frac{\partial \overline{WT}}{\partial t}\right)\right\}, \tag{1.13}$$

where G is that part of β which vanishes when the time variations of $\overline{WT}$ vanish. G is perhaps important in the study of the onset of the aperiodic 'turbulent' convection, but since it will soon disappear from this problem its explicit form need not be given here.

At this point it is of value to non-dimensionalize the physical quantities appearing in the equations. This will simplify the mathematical manipulations and produce the pertinent physical parameters in the problem. Let

$$\mathbf{V} \equiv (\kappa/d)\mathbf{V}', \quad T \equiv (\nu\kappa/\gamma d^3)T', \quad (x,y,z) \equiv d(x',y',z'), \quad t \equiv (d^2/\kappa)t'. \tag{1.14}$$

Having made this change all primes will now be dropped. *In the remainder of this work all unprimed quantities are non-dimensional unless it is otherwise stated.* Using (1.13) and (1.14), equations (1.6), (1.8) and (1.11) become

$$\tfrac{1}{2}\partial(\overline{T^2})_m/\partial t = \lambda(\overline{WT})_m + \{(\overline{WT})_m^2 - (\overline{WT^2})_m + (\overline{WTG})_m\} - (\overline{\nabla T.\nabla T})_m \tag{1.15}$$

$$(\sigma^{-1}\,\partial/\partial t - \nabla^2)\nabla^2 W = \nabla_1^2\, T + \sigma^{-1}L(\mathbf{M}) \tag{1.16}$$

$$(\partial/\partial t - \nabla^2)(\sigma^{-1}\,\partial/\partial t - \nabla^2)\nabla^2 W - \lambda\nabla_1^2\, W - \{(\overline{WT})_m - \overline{WT} + G\}\nabla_1^2\, W = -\nabla_1^2 h + \sigma^{-1}(\partial/\partial t - \nabla^2)L(\mathbf{M}) \tag{1.17}$$

where $\sigma \equiv \nu/\kappa$ is the Prandtl number, and λ is the previously defined Rayleigh number. The non-dimensional forms of equations (1.9) and (1.10) will be used later in the analysis but are not needed immediately.

In the usual studies of disturbances of the state of steady conduction, one seeks solutions of the form

$$\left.\begin{aligned} W &= \epsilon W_0 + \epsilon^2 W_1 + \epsilon^3 W_2 + \ldots, \\ T &= \epsilon T_0 + \epsilon^2 T_1 + \epsilon^3 T_2 + \ldots, \end{aligned}\right\} \tag{1.18}$$

with similar expansions for the other variables, where ϵ is a constant parameter. It is then required that expressions (1.18) satisfy the complete equations of motion for all values of ϵ less than some maximum ϵ. The coefficients of each power of ϵ generated by substituting the expressions (1.18) into the equations of motion must vanish individually and the resulting series for each of the variables must converge if relations (1.18) are to represent a satisfactory solution to the problem. Stability theory is concerned with the solutions of the first-order equation only. For first-order solutions to be complete one must require that ϵ be proportional to the amplitude of the disturbance and that this amplitude be infinitesimal. In this finite amplitude study we will find that a similar identification of ϵ and amplitude is necessary if (1.18) is to represent a solution*.

An important point in this expansion remains to be discussed. When the actual forms for the W_i and T_i are substituted into (1.15), one finds an explicit relation between ϵ and λ. Thus

$$\lambda = \lambda_0 + \epsilon\lambda_1 + \epsilon^2\lambda_2 + \dots, \tag{1.19}$$

where the numbers λ_i are integrals of functions of W_i and T_i. Such a relation between the amplitude and the physical parameter is to be expected. One might have preferred the amplitude to be given as a power series in λ since λ is actually the variable which can be controlled. However, (1.19) is a natural consequence of the expansion (1.18). Explicit determination of the λ_i is discussed in the following paragraphs.

The sequence of linear inhomogeneous equations in powers of ϵ generated by inserting equations (1.18) and (1.19) into equation (1.17) is

$$\left.\begin{aligned}
\mathscr{L}(W_0) &\equiv (\partial/\partial t - \nabla^2)(\sigma^{-1}\,\partial/\partial t - \nabla^2)\nabla^2 W_0 - \lambda_0 \nabla_1^2\, W_0 = 0,\\
\mathscr{L}(W_1) &= \lambda_1\, \nabla_1^2\, W_0 - \nabla_1^2 h_{00} + \sigma^{-1}(\partial/\partial t - \nabla^2)L_{00},\\
\mathscr{L}(W_2) &= \lambda_1\, \nabla_1^2 W_1 + \lambda_2\, \nabla_1^2 W_0 + \{(\overline{W_0\, T_0})_m - \overline{W_0\, T_0} + G_{00}\}\nabla_1^2\, W_0 -\\
&\qquad - \nabla_1^2(h_{01} + h_{10}) + \sigma^{-1}(\partial/\partial t - \nabla^2)(L_{01} + L_{10}),\\
\mathscr{L}(W_i) &= \sum_0^{n-1} \lambda_{i-n} \nabla_1^2\, W_n + \sum_{n+l=0}^{i-2} \{(\overline{W_n T_l})_m - \overline{W_n T_l} + G_{nl}\}\nabla_1^2\, W_{i-(n+l+2)} +\\
&\qquad + \sum_{n=0}^{i-1} \{-\nabla_1^2 h_{n,i-(n+1)} + \sigma^{-1}[\partial/\partial t - \nabla^2]L_{n,i-(n+1)},
\end{aligned}\right\} \tag{1.20}$$

where the subscripts n, l on the non-linear terms h, L and G mean that W_n or $\mathbf{V}_n$ and T_l are to be substituted for W or $\mathbf{V}$ and T in these terms. The T_i can be determined from the W_i by the auxiliary equations derived from (1.16)

$$\left.\begin{aligned}
\nabla_1^2\, T_0 &= (\sigma^{-1}\,\partial/\partial t - \nabla^2)W_0,\\
\nabla_1^2\, T_i &= (\sigma^{-1}\,\partial/\partial t - \nabla^2)\nabla^2 W_i - \sigma^{-1} \sum_0^{i-1} L_{n,i-(n+1)},
\end{aligned}\right\} \tag{1.21}$$

with similar expansions for U_i and V_i. Each of the W_i must satisfy the boundary conditions for the vertical velocity, each of the T_i the boundary conditions for the temperature fluctuations. The various boundary conditions of the problem will be discussed in the next section.

The first of equations (1.20) is the classical Rayleigh stability equation, and $\mathscr{L}$ can be termed the linear constant-coefficient Rayleigh operator. Assume tentatively that it is solved for W_0 (normalized) and λ_0, and that from the auxiliary equations T_0, U_0, V_0 are also known. The second of equations (1.20) is a linear inhomogeneous equation for W_1. The inhomogeneous terms depend on the known forms of W_0, T_0, U_0, V_0 and the unknown

* V. S. Sorokin (1954) attempted to solve the second order equations by assuming that $\epsilon \sim (\sqrt{\lambda} - \sqrt{\lambda_0})^{1/2}$. His neglect of the important terms containing β/β_m in the basic equations prevented him from obtaining any quantitative finite-amplitude results.

number λ_1. The solution of this equation for W_1 will be the sum of a homogeneous solution plus the particular integral generated by these inhomogeneous terms. Three difficulties arise at this point. (1) The part of the homogeneous term which has the form of W_0 will produce a secular (non-periodic) response in the particular integral—a solution which has no physical counterpart; (2) that part of the homogeneous term which satisfies the boundary conditions will contain an arbitrary constant; (3) a part of the inhomogeneous term may satisfy $\mathscr{L}(W) = 0$ but not the boundary conditions (i.e. it will differ from W_0) and will also give rise to a secular response. The last occurs nowhere in our analysis because the inhomogeneous terms themselves always satisfy the boundary condition. The first two complications require that a method be adopted which resolves the indeterminacy and which makes the ϵ-equations a soluble iterative sequence.

We have found only one such method*. It is to require (1) that the λ_i be evaluated so as to eliminate the 'resonant' inhomogeneous terms, for example

$$\lambda_1(\overline{W_0 \nabla_1^2 W_0})_m = (\overline{W_0 \nabla_1^2 h_{00}})_m - \sigma^{-1}\{\overline{W_0(\partial/\partial t - \nabla^2)L_{00}}\}_m, \qquad (1.22)$$

and (2) that $(\overline{W_0 W})_m = \epsilon$, i.e. that ϵ be proportional to the amplitude of that part of W which has the form W_0. Then from (1.18) all W_i ($i > 0$) must be orthogonal to W_0, though not necessarily orthogonal to each other. The latter requirement removes the arbitrary constants of the homogeneous parts of the W_i equations.

To eliminate the resonant terms in the second-order equation, we must have

$$\begin{aligned}\lambda_2(\overline{W_0 \nabla_1^2 W_0})_m^{-1} = & -\lambda_1(\overline{W_0 \nabla_1^2 W_1})_m - \\ & -\{\overline{W_0[(\overline{W_0 T_0})_m - \overline{W_0 T_0} + G_{00}]\nabla_1^2 W_0}\}_m + \\ & + [\overline{W_0 \nabla_1^2(h_{01} + h_{10})}]_m - \sigma^{-1}[\overline{W_0(\partial/\partial t - \nabla^2)(L_{01} + L_{10})}]_m. \end{aligned} \qquad (1.23)$$

Higher λ_i generated from (1.20) by the first requirement above are

$$\begin{aligned}\lambda_i = (\overline{W_0 \nabla_1^2 W_0})_m^{-1}\Big\{ & -\sum_{n=0}^{i-1}\lambda_{i-(n+1)}(\overline{W_0 \nabla_1^2 W_n})_m + (\overline{W_0 \nabla_1^2 h_{n,i-n+1}})_m - \\ & - \sigma^{-1}[\overline{W_0(\partial/\partial t - \nabla^2)L_{n,i-(n+1)}}]_m - \\ & - \sum_{n+l=0}^{i-2} [\overline{\{(\overline{W_n T_l})_m - \overline{W_n T_l} + G_{nl}\} W_0 \nabla_1^2 W_{i-(n+l+2)}}]_m\Big\}. \end{aligned} \qquad (1.24)$$

A formal technique for determining the amplitude and distortion of a disturbance as a function of the external parameters is now complete. Can one determine the range of λ in which a series solution terminated after a term ϵ^i is a good approximation to the whole series for W? The best we can do to test the validity of a limited solution is to compare the

* Recently we have learned that an analogous method was proposed by Lindstedt (1883) to obviate similar difficulties which arise in problems of celestial mechanics.

amplitude of W_i to ϵ, the amplitude of W_0. If this ratio exceeds 0·1, for example, we can be fairly sure that an additional function W_{i+1} must be computed to keep errors in distortion below 10%. However, the important energy and heat transport terms are proportional to products of W and T. Hence the error in these terms will be much smaller than the error of W and T themselves. Examples of determination of the errors in limited solutions are given in the following section.

The proposed technique cannot answer all the explicit questions raised in the Introduction. If a solution to the stability problem, W_0, is independent of time, then none of the equations (1.20) can lead to time dependence. If there are many solutions to the stability problem, then the equations (1.20) give as many answers. To determine which of these answers is realized in an experiment we must separately investigate their 'relative stability'.

Therefore, we do not present this technique as a unique statement of the finite amplitude problem. Several alternative approaches which also answer some but not all of our questions are discussed in the Conclusion.

The following section discusses the consequences of this ϵ-expansion with generating functions based on the simplest solution of the Rayleigh problem.

2. Analysis for the two-dimensional case with free surfaces

The first step in a determination of the effects of finite amplitude in equation (1.20) is the resolution of the stability problem. Pellew & Southwell (1940) have made a most comprehensive study, which will be only briefly outlined here. They investigate the first of equations (1.20), $\mathscr{L}(W_0) = 0$, to determine W_0 and λ_0 for three different sets of boundary conditions on the velocities. In every case the boundaries are perfect heat conductors, that is, the fluctuation temperature vanishes there. The conditions at a free boundary are

$$W = \partial^2 W/\partial z^2 = T = 0, \tag{2.1}$$

and are a consequence of the divergence relation (1.3) when the boundary cannot support a stress. The conditions at a rigid boundary are

$$W = \partial W/\partial z = T = 0, \tag{2.2}$$

and are a consequence of the divergence relation when all velocity components must vanish at the boundary.

The assumption made in solving the Rayleigh problem is that the field of motion is separable, i.e.

$$\left.\begin{aligned} W_0 &= \phi(x,y)F(z)G(t), \\ \text{and} \qquad \nabla_1^2\phi(x,y) &= -\alpha^2\pi^2\phi(x,y), \end{aligned}\right\} \tag{2.3}$$

where the separation parameter α is the effective wave-number of the disturbance in the horizontal plane. (In what follows we treat $\phi(x,y)$ as a normalized function.) Pellew & Southwell have shown that this assumption is justified only if the horizontal plan-form of the motion consists of regular 'close-packed cells' at whose lateral boundaries the normal derivatives of the temperature and velocity fluctuations vanish. The plan-forms which satisfy these requirements are the two-dimensional 'roll', the general rectangle, the equilateral triangle and the hexagon. Hence a characteristic value λ_0 will have associated with it an infinite number of possible characteristic functions, all with the same α. Pellew & Southwell correctly suggest that this degeneracy in the first-order theory is removed by 'higher-order' effects, though they were apparently under the impression that the hexagonal plan-form was the preferred motion in all experiments.

Another important contribution of Pellew & Southwell to this first-order problem is their proof that all oscillatory solutions decay. Hence maintained convection is initiated as a steady motion and "limiting conditions of stability are in fact obtained when all time variations (in $\mathscr{L}(W_0) = 0$) are made zero".

Therefore, with the use of (2.3), the equation $\mathscr{L}(W_0) = 0$ becomes

$$(\partial^2/\partial z^2 - \alpha^2\pi^2)^3 W_0 + \lambda_0 \alpha^2\pi^2 W_0 = 0, \tag{2.4}$$

which is a linear sixth-order equation with constant coefficients. Its general solution is

$$W_0 = \phi(x,y) \sum_{i=1}^{3} (A_i \cosh 2\mu_i z + B_i \sinh 2\mu_i z), \tag{2.5}$$

where A_i and B_i are arbitrary constants and

$$4\mu_i^2 \equiv \alpha^2\pi^2\{1-(\lambda_0/\alpha^2\pi^2)^{1/3}\omega_i\}, \tag{2.6}$$

where the ω_i are the three cube-roots of unity.

Application of the conditions for two free boundaries to (2.5) and (2.6) generates the characteristic functions

$$W_0 = A\,\phi(x,y)\sin n\pi z, \tag{2.7}$$

where n is an arbitrary integer, and the characteristic values are

$$\lambda_{0n} = \pi^4(n^2+\alpha^2)^3/\alpha^2. \tag{2.8}$$

The lowest value of λ_{0n} is $27\pi^4/4$, and it occurs for $\alpha^2 = \frac{1}{2}$ and $n = 1$. The case $n = 1$ is our primary concern, since, before the modes corresponding to higher values of n can occur, the mode corresponding to $n = 1$ has grown to finite amplitude, markedly altering the basic temperature field.

For two rigid boundaries the characteristic function becomes

$$W_0 = \phi(x,y) \sum_{1}^{3} A_i \cosh 2\mu_i(z-\tfrac{1}{2}), \tag{2.9}$$

where $2z = \pm 1$ at the boundaries and where A_2 and A_3 are related to A_1 by the conditions

$$\sum_{1}^{3} A_i \cosh \mu_i = 0, \qquad \sum_{1}^{3} \mu_i A_i \sinh \mu_i = 0. \tag{2.10}$$

The μ_i are found from (2.6) once λ_0 and α are known. The relation between λ_0 and α has been determined numerically by Pellew & Southwell (1940, p. 377). The lowest value of λ_0 is 1707·8 for $\alpha^2 = 3{\cdot}13/\pi$.

For one rigid and one free boundary the first characteristic function is the second, or 'odd' mode of (2.9) with $2z = 1$ at the rigid boundary and $2z = 0$ at the free boundary. Pellew & Southwell found the minimum eigenvalue to be $\lambda_0 = 1100{\cdot}65$ for $\alpha^2 = 2{\cdot}68/\pi$.

The primary conclusion drawn from the first-order equations is that for $\lambda < \lambda_0$ all infinitesimal disturbances to the purely conductive state decay in time. What will be the equilibrium amplitude of the preferred disturbances for $\lambda \geqslant \lambda_0$?

Finite amplitude solution for rolls

In the remainder of this section we will study the solutions generated by (1.20) for the very simplest case, that of the two-dimensional roll with two free boundaries. Many, but not all, of the properties of the other solutions to this problem are exhibited in this case. However, it must be borne in mind that neither the two free boundaries nor the roll can be realized in practice.

A complete solution to the first-order equations for strip plan-form and two free boundaries is

$$\left.\begin{aligned} W_0 &= 2\cos\pi\alpha x\sin\pi z,\\ T_0 &= (1+\alpha^2)^2(2\pi^2/\alpha^2)\cos\pi\alpha x\sin\pi z,\\ U_0 &= -(2/\alpha)\sin\pi\alpha x\cos\pi z, \qquad V_0 = 0,\\ \lambda_0 &= \pi^4(1+\alpha^2)^3/\alpha^2, \end{aligned}\right\} \tag{2.11}$$

from (1.20), (1.21), (1.9), (1·10) and (2.1).

As noted previously the lowest value of λ_0 is $27\pi^4/4$ and occurs when $\alpha^2 = \frac{1}{2}$. Before solving the second-order equation for W_1, the quantities h_{00}, L_{00} and λ_1 must be determined from (2.11). Now

$$h_{00} \equiv \mathbf{V}_0 . \nabla T_0 - \frac{\partial}{\partial z}\,(\overline{W_0 T_0}) = \frac{4\pi^2(1+\alpha^2)^2}{\alpha^2}\left[\tfrac{1}{2}\pi\sin 2\pi z - \frac{\partial}{\partial z}\,(\tfrac{1}{2}\sin^2\pi z)\right] = 0,$$

independently of the choice of α. Also

$$L_{00} \equiv \frac{\partial^2}{\partial x\partial z}(\mathbf{V}_0 . \nabla U_0) + \frac{\partial^2}{\partial y\partial z}(\mathbf{V}_0 . \nabla V_0) - \nabla_1^2\mathbf{V}_0 . \nabla W_0 = 0.$$

Hence from (1.22) $\lambda_1 = 0$. We shall find that $\lambda_1 = 0$ for all symmetric solutions to the first-order equations. However, the conclusion that $h_{00} = L_{00} = 0$ is true only for rolls and only with two free boundaries. This fortuitous vanishing of all zero-average advection by the first-order solutions leads to $\mathscr{L}(W_1) = 0$; hence $W_1 = U_1 = T_1 = 0$ from (1.20) and (1.21). Therefore $h_{01} = h_{10} = L_{01} = L_{10} = 0$, and the third-order equation of (1.20) becomes

$$\mathscr{L}(W_2) = \lambda_2\nabla_1^2 W_0 + \{(\overline{W_0 T_0})_m - \overline{W_0 T_0}\}\nabla_1^2 W_0, \tag{2.12}$$

where λ_2 is given by

$$\lambda_2(\overline{W_0\nabla_1^2 W_0})_m = -[\{(\overline{W_0 T_0})_m - \overline{W_0 T_0}\}\overline{W_0\nabla_1^2 W_0}]_m \tag{2.13}$$

from (1.23). From (2.11) and (2.13)

$$\lambda_2 = \tfrac{1}{2}\pi^2(1+\alpha^2)^2/\alpha^2. \tag{2.14}$$

Therefore (2.12) becomes

$$\nabla^6 W_2 - \lambda_0 \nabla_1^2 W_2 = -\pi^4(1+\alpha^2)^2 \cos \pi\alpha x \sin 3\pi z. \qquad (2.15)$$

Before investigating the distortions of the roll form generated by (2.15), the implications of our first finite-amplitude results, equation (2.14), will be studied. From (1.19), (1.18) and (2.11), to this third-order approximation, $\epsilon^2 = (\lambda - \lambda_0)/\lambda_2$ and

$$(\overline{WT})_m = \epsilon^2(\overline{W_0 T_0})_m = 2(\lambda - \lambda_0). \qquad (2.16)$$

Also, from (1.13) in its non-dimensional form,

$$\beta/\beta_m = 1 + 2(1 - \lambda_0/\lambda)\cos 2\pi z. \qquad (2.17)$$

In (2.17) the distortion of the mean temperature gradient β/β_m is determined by products of zero-order functions only and requires that for $\lambda > 2\lambda_0$ the gradient should become negative in the mid-regions of the fluid. Equation (2.16) asserts a linear relation between the convective heat transport and λ. (Note that this relation is independent of σ.) That the observed heat transport shows just such a linear relation from λ_0 to the second transition at $10\lambda_0$ suggests that this third approximation provides an adequate description of the field of motion throughout the entire range of steady convection. However, this is not the case as we shall now establish by the investigation of higher order terms.

λ_4-approximation

The solution of (2.15) for W_2 is

$$W_2 = C_W \cos \pi\alpha x \sin 3\pi z, \qquad (2.18)$$

where

$$C_W = \frac{\pi^2(1+\alpha^2)^2}{\pi^4(3^2+\alpha^2)^3 - \lambda_0 \alpha^2}.$$

Hence from (1.21) and (1.9)

$$T_2 = C_T \cos \pi\alpha x \sin 3\pi z, \quad U_2 = -(3/\alpha) C_W \sin \pi\alpha x \cos 3\pi z, \qquad (2.19)$$

where

$$C_T = (\pi/\alpha)^2 (3^2+\alpha^2)^2 C_W.$$

Therefore the first distortion of a finite-amplitude roll does not affect the horizontal plan-form, but only the vertical structure. The addition of the term $\sin 3\pi z$ to the first-order solution increases the amplitude of W and T near the boundaries in keeping with the increased gradients near the boundaries (see equation (2.17)).

This rather straightforward iterative analysis will now be extended to determine λ_3, λ_4, λ_5 and λ_6 and the accompanying W_i, T_i, U_i.

From (2.11) and (2.19) one finds that

$$\nabla_1^2(h_{02}+h_{20}) = C_1(\alpha)\cos 2\pi\alpha x(2\sin 2\pi z + \sin 4\pi z)$$

and $$\sigma^{-1}\nabla^2(L_{02}+L_{20}) = \sigma^{-1}C_2(\alpha)\cos 2\pi\alpha x(2\sin 2\pi z + 3\sin 4\pi z). \qquad (2.20)$$

Since W_0 is orthogonal to the expressions (2.20), and since $\overline{W_0 T_1} = \overline{W_1 T_0} = 0$, then $\lambda_3 = 0$. Therefore

$$\mathscr{L}(W_3) = \cos 2\pi\alpha x[C_3(\alpha,\sigma)\sin 2\pi z + C_4(\alpha,\sigma)\sin 4\pi z]. \qquad (2.21)$$

W_3 and T_3 are of the same form as $\mathscr{L}(W_3)$, but with different coefficients $C(\alpha,\sigma)$, and

$$U_3 = \sin 2\pi\alpha x[C_5(\alpha,\sigma)\cos 2\pi z + C_6(\alpha,\sigma)\cos 4\pi z].$$

In order to determine λ_4 one must construct h_{03}, h_{30}, L_{03} and L_{30} from (2.11) and (2.21). This leads to

$$\nabla_1^2(h_{03}+h_{30}) = \cos\pi\alpha x(D_1\sin 3\pi z + D_2\sin 5\pi z) + \\ + \cos 3\pi\alpha x(D_3\sin\pi z + D_4\sin 3\pi z + D_5\sin 5\pi z), \qquad (2.22)$$

with an identical form for $\nabla^2(L_{03}+L_{30})$ differing only in the coefficients $D(\alpha,\sigma)$. Therefore

$$[\overline{W_0\nabla_1^2(h_{03}+h_{30})}]_m = [\overline{W_0\nabla^2(L_{03}+L_{30})}]_m = 0 \qquad (2.23)$$

and

$$\lambda_4 = \frac{[(\overline{W_0 T_2} + \overline{W_2 T_0})\overline{W_0\nabla_1^2 W_0}]_m + (\overline{W_0 T_0}\,\overline{W_0\nabla_1^2 W_2})_m}{(\overline{W_0\nabla_1^2 W_0})_m}$$
$$= -\tfrac{1}{4}C_T + 2\pi^2(1+\alpha^2)^2\alpha^{-2}C_W. \qquad (2.24)$$

Since λ_4 is an integral of zero-order and second-order functions only, it is not dependent upon σ. Hence, to the fifth order, σ does not influence the amplitude of the convection. Indeed the largest part of the σ-dependent inhomogeneous term in the equation for W_4, namely, $\sigma^{-1}\nabla^2(L_{03}+L_{30})$, is smaller than the corresponding term in $\nabla_1^2(h_{03}+h_{30})$ by a factor $\alpha^2/[2\pi^2(1+\alpha)^2\sigma]$, $\doteqdot 10^{-2}/\sigma$. Hence in the determination of W_4 and λ_6 these higher-order momentum advection terms will be neglected.

λ_6-approximation

Proceeding as above one finds that

$$W_4 = \cos\pi\alpha(E_1\sin 3\pi z + E_2\sin 5\pi z) + \\ + \cos 3\pi\alpha(E_3\sin\pi z + E_4\sin 3\pi z + E_5\sin 5\pi z), \qquad (2.25)$$

with a similar form for T_4 where $E = E(\alpha,\sigma)$.

As for previous λ_i with odd i, $\lambda_5 = 0$ since $\overline{W_0 W_3} = \overline{W_0 T_3} = \overline{W_3 T_0} = 0$ and $[\overline{W_2(h_{20}+h_{02})}]_m = 0$. Therefore

$$\lambda_6 = (\overline{W_0\nabla_1^2 W_0})_m^{-1}\{(\overline{W_0 T_4} + \overline{W_4 T_0})(W_0\nabla_1^2 W_0) + (\overline{W_2 T_0} + \overline{W_0 T_2})\overline{W_0\nabla_1^2 W_2} + \\ + \overline{W_0 T_0}\,\overline{W_0\nabla_1^2 W_4} - (\alpha/\pi)^2(1+\alpha^2)^{-2}[T_2\nabla_1^2(h_{03}+h_{30}) + \\ + T_3\nabla_1^2(h_{02}+h_{20})]\}_m, \qquad (2.26)$$

where use has been made of the facts that $(\overline{Th})_m \doteqdot 0$ and $T_0 = N_0 W_0$ in order to simplify the last term.

Here for the first time the Prandtl number can have an effect upon amplitude, since the coefficients in W_3 and W_4 are functions of σ.

We are now able to investigate the ranges of λ in which these various solutions W_2, W_3, W_4 first become significant contributors to the field of motion.

For the initially unstable disturbance $\lambda_0 = 27\pi^4/4$ and $\alpha^2 = \frac{1}{2}$ and from (2.14), (2.18), (2.19) and (2.24) $\lambda_2 = 9\pi^2/4$ and $\lambda_4 = -0{\cdot}1248 \doteqdot -\frac{1}{8}$. Then, to this λ_4-approximation, from (1.19)

$$\epsilon^2 = \frac{1}{2}\left\{-\frac{\lambda_2}{\lambda_4} + \left[\left(\frac{\lambda_2}{\lambda_4}\right)^2 + 4\frac{(\lambda-\lambda_0)}{\lambda_4}\right]^{1/2}\right\}. \tag{2.27}$$

Equation (2.27) tells one that ϵ^2 becomes imaginary when

$$\lambda-\lambda_0 = -\tfrac{1}{4}\lambda_2^2/\lambda_4 \doteqdot 1{\cdot}5\lambda_0. \tag{2.28}$$

Hence the range of λ in which this λ_4-approximation is valid is certainly less than $1{\cdot}5\lambda_0$, while the range in which the previous λ_2-approximation is valid must be smaller still. When $\lambda-\lambda_0 = 1{\cdot}5\lambda_0$, $\epsilon^2 = -\frac{1}{2}\lambda_2/\lambda_4 = 2(\lambda-\lambda_0)/\lambda_2$ from (2.27) and (2.28), and ϵ^2 is twice as great as the amplitude predicted in (2.16) for the λ_2-approximation. Therefore the hope that the λ_2-approximation would be valid throughout the laminar convection range is unjustified. It is now clear that the distortion of the initial disturbance must play a significant part in determining the observed heat transport.

The λ_6-computation from (2.26) though tedious, has proved of considerable value in explaining both the observed relation between λ and $(\overline{WT})_m$ and the nature of the ϵ-expansion. Since λ_6 is a function of σ, the various coefficients, $C(\alpha,\sigma)$, $D(\alpha,\sigma)$, $E(\alpha,\sigma)$, entering into it have been computed for $\alpha^2 = \frac{1}{2}$ at three values of σ. For $\sigma = 0{\cdot}8$, a value appropriate for a gas, $\lambda_6 = 2{\cdot}31\times10^{-3}$. For $\sigma = 8$, a value appropriate for water, $\lambda_6 = 2{\cdot}30\times10^{-3}$. For $\sigma=\infty$, $\lambda_6 = 2{\cdot}29\times10^{-3}$. It would be necessary to include the small neglected σ-term in W_4 in order to decide whether the effect of σ on heat transport is just very small or whether it vanishes completely. We will return to the effects of σ in the following section. Here we will study amplitudes and distortions when $\lambda_6 = 2{\cdot}30\times10^{-3}$ corresponding to the value of σ for water.

In order to determine ϵ as a function of λ, to the λ_6-approximation, from (1.19) one must solve the cubic equation

$$(\epsilon^2)^3\lambda_6 + (\epsilon^2)^2\lambda_4 + \epsilon^2\lambda_2 + \lambda_0 - \lambda = 0, \tag{2.29}$$

for $\lambda_0 = 27\pi^4/4$, $\lambda_2 = 9\pi^2/4$, $\lambda_4 = -\frac{1}{8}$, $\lambda_6 = 2{\cdot}30\times10^{-3}$. The heat transport to this same approximation is

$$(\overline{WT})_m = \epsilon^2(\overline{W_0\,T_0})_m[1+\epsilon^4(\overline{W_2\,T_2})_m/(\overline{W_0T_0})_m]$$
$$= \epsilon^2(\overline{W_0\,T_0})_m(1+7{\cdot}15\times10^{-7}\epsilon^4) \tag{2.30}$$

from (1.18), (2.18) and (2.19). In figure 2 we have plotted the total heat transport, divided by its value at λ_0 against λ/λ_0 for the λ_6-approximation, for the λ_4-approximation from (2.27), and for the λ_2-approximation from (2.16). The two additional curves labelled α^2 and λ_6 (*ZANL*) will be discussed shortly.

Effects of λ_6 and zero-average non-linear terms

Perhaps the most interesting aspect of the curve for λ_6 is that it has brought the heat transport back very nearly to the λ_2-curve in the range $\lambda_0 \leqslant \lambda \leqslant 3\lambda_0$. This alternating character of the various approximations to λ makes the observed linear relation between heat transport and λ more understandable. One can anticipate that the λ_8-curve, like the λ_4-curve, will diverge to the right at some λ greater than $3\lambda_0$, while the λ_{10}-curve,

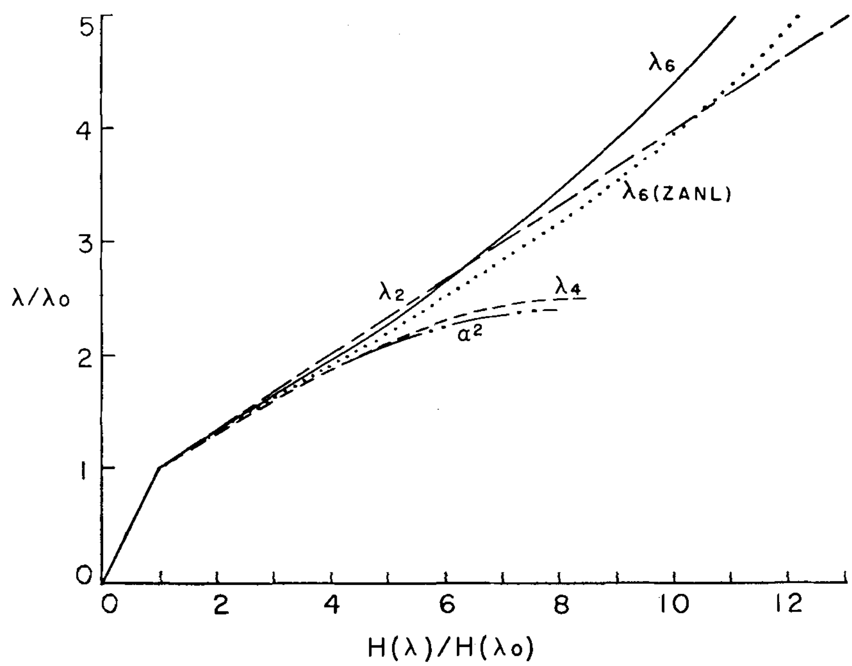

Figure 2. Heat transport *vs* Rayleigh number for the second, fourth and sixth approximations, for the fourth approximation with variable α^2, and for the sixth approximation without zero-average non-linear terms.

like the λ_6-curve, will be more nearly parallel to that for λ_2. This alternation also suggests that a more satisfactory formulation of our problem should be sought permitting an expansion $\lambda = \lambda(\epsilon)$ in some monotonic sequence.

The percentage error of the λ_6-curve at various λ can be estimated by comparing the magnitude of the various terms in the expansion of $\overline{WT}$ (or equivalently of β/β_m). From (1.18), (1.13), and the preceding equations for W_i and T_i,

$$\begin{aligned}\overline{WT} &= \epsilon^2\{\overline{W_0 T_0} + \epsilon^2(\overline{W_0 T_2} + \overline{W_2 T_0}) + \epsilon^4(\overline{W_0 T_4} + \overline{W_2 T_2} + \overline{W_4 T_0}) + \ldots\} \\ &= \epsilon^2(\overline{W_0 T_0})_m\{2\sin^2 \pi z + \epsilon^2(1{\cdot}098 \times 10^{-2})\sin \pi z \sin 3\pi z + \\ &\quad + \epsilon^4[1{\cdot}43 \times 10^{-6} \sin^2 3\pi z - 2{\cdot}22 \times 10^{-5} \sin \pi z \sin 3\pi z + \\ &\quad + 2{\cdot}38 \times 10^{-5} \sin \pi z \sin 5\pi z\}. \qquad (2.31)\end{aligned}$$

Therefore

$$\frac{\beta}{\beta_m} = 1 + \frac{(\overline{WT})_m - \overline{WT}}{\lambda}$$
$$= 1 + \epsilon^2(\overline{W_0 T_0})_m \lambda^{-1}\{(1 - 5{\cdot}49 \times 10^{-3}\epsilon^2 + 1{\cdot}11 \times 10^{-5}\epsilon^4)\cos 2\pi z +$$
$$+ (5{\cdot}49 \times 10^{-3}\epsilon^2 - 2{\cdot}3 \times 10^{-5}\epsilon^4)\cos 4\pi z + 1{\cdot}26 \times 10^{-5}\epsilon^4 \cos 6\pi z + \ldots\}. \quad (2.32)$$

For $\lambda = 3\lambda_0$, equation (2.29) has the root $\epsilon^2 = 57{\cdot}7$, whence

$$\beta/\beta_m = 1 + 1{\cdot}301\,[0{\cdot}801 \cos 2\pi z + 0{\cdot}160 \cos 4\pi z + 0{\cdot}041 \cos 6\pi z + \ldots], \quad (2.33)$$

suggesting that at this value of λ the error due to neglected terms is less than 2%. This error is of course much smaller at smaller λ, but will rise rapidly for $\lambda > 3\lambda_0$.

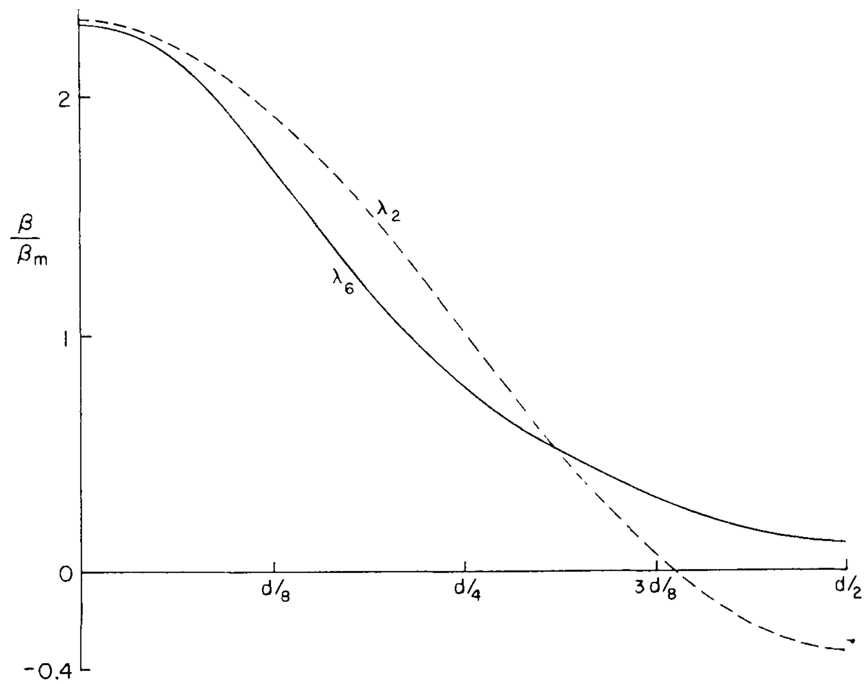

Figure 3. Comparison of the mean temperature gradients predicted by the second and sixth approximations for free rolls at $\lambda = 3\lambda_0$.

An interesting consequence of relation (2.32) is depicted in figure 3. The curves β/β_m for λ_2 from (2.17) and for λ_6 from (2.33) at $\lambda = 3\lambda_0$ are drawn from a boundary to the middle of the fluid. Despite the negligible change in heat transport (given by the gradients at the boundary), the distortion of the finite amplitude disturbance has prevented the gradient from becoming negative. This is probably a property of the convection at all λ, but no general proof that $\beta/\beta_m \geqslant 0$ has yet been found.

Another matter of some interest is the effect of the zero-average non-linear terms h and $L(\mathbf{M})$ on the transport of heat. One can easily recompute

λ_2, λ_4, and λ_6, neglecting h and $L(\mathbf{M})$, and compare the resulting transport with that found in (2.29) and (2.30). From (2.13) and (2.24) it is seen that λ_2 and λ_4 are unchanged. However, a redetermination of W_4 and T_4 leads to $\lambda_6 = 1{\cdot}44 \times 10^{-3}$ from (2.26). A new solution of the cubic (2.29) with this value of λ_6 determines the heat transport curve labelled $\lambda_6(ZANL)$ in figure 2. One concludes that, to this approximation, neglect of the zero-average non-linear terms increases the amplitude of the predicted heat transport.

Amplitude effects on the horizontal wave-number

Up to this point in the calculations we have considered only the growth process of the free roll when $\alpha^2 = \frac{1}{2}$, its optimum value for a lowest λ_0. However, it is improbable that the distortions of the growing disturbance are not accompanied by a change in the basic horizontal wave-number. Indeed the expressions for λ_0, λ_2, and λ_4 (see (2.11), (2.14) and (2.24)) have been written as explicit functions of α^2 so that we may investigate the effects of changing α on the heat transport.

We have as yet no criterion to select the physically realized finite amplitude solution from the manifold of steady solutions permitted by the equations. In first-order theory the criterion is to find the lowest value of λ_0, for this disturbance is the 'first' to grow. But for λ greater than this lowest λ_0 a variety of disturbances of different α^2 could grow, each corresponding to different $\lambda > \lambda_{\min}$. In a following section we shall establish a 'relative stability' criterion to select the realized solution. Here, we shall investigate one extreme, namely, that solution which leads to the maximum heat transport.

To the λ_2-approximation the heat transport is a maximum for minimum λ_0, i.e. at $\alpha^2 = \frac{1}{2}$, for all λ. However from (1.19), (2.14) and (2.24), to the λ_4-approximation,

$$(\overline{WT})_m = \lambda_2\left\{-\frac{\lambda_2}{\lambda_4} - \left[\left(\frac{\lambda_2}{\lambda_4}\right)^2 - \frac{4(\lambda-\lambda_0)}{\lambda_4}\right]^{1/2}\right\}, \tag{2.34}$$

where

$$\lambda_0 = \frac{\pi^4(1+\alpha^2)^3}{\alpha^2}, \quad \lambda_2 = \frac{\pi^2(1+\alpha^2)^2}{2\alpha^2}, \quad \lambda_4 = -\frac{(1+\alpha^2)^2}{4\alpha^2}\frac{(3^2+\alpha^2)^2+2(1+\alpha^2)^2}{(3^2+\alpha^2)^3-(1+\alpha^2)^3}.$$

We seek that $\alpha^2 = \alpha^2(\lambda)$ which leads to a maximum $(\overline{WT})_m$. From (2.34) this extreme relation is

$$\frac{\partial}{\partial\alpha^2}(\overline{WT})_m = 0 = \frac{\partial K}{\partial\alpha^2} - \frac{2K}{2(\lambda-\lambda_0)-(\overline{WT})_m}\frac{\partial\lambda_0}{\partial\alpha^2}, \tag{2.35}$$

where $K \equiv \lambda_2^2/|\lambda_4|$. An explicit solution for α^2 first will be sought for λ close to $(\lambda_0)_{\min}$ where α^2 will be close to $\frac{1}{2}$. We let $\alpha^2 = \frac{1}{2}+\Delta$; then

$$K = 6\lambda_c(1-a\Delta+b\Delta^2+\ldots) \tag{2.36}$$

and

$$\lambda_0 = \lambda_c(1+\tfrac{4}{3}\Delta^2+\ldots),$$

where $\lambda_c \equiv (\lambda_0)_{\min}$, $a \doteqdot 0{\cdot}621$ and $b \doteqdot 1{\cdot}782$ from (2.34). Hence

$$(\overline{WT})_m = 6\lambda_c[1-(1-\tfrac{2}{3}x)^{1/2}+A\Delta-B\Delta^2+\ldots], \tag{2.37}$$

where

$$A = a\left\{\frac{1-\frac{1}{3}x}{(1-\frac{2}{3}x)^{1/2}}-1\right\}, \qquad x \equiv \frac{\lambda-\lambda_c}{\lambda},$$

$$B = \frac{b+\frac{1}{2}a^2+\frac{4}{9}-\frac{1}{3}bx}{(1-\frac{2}{3}x)^{1/2}} - \frac{a^2}{2}\frac{(1-\frac{1}{3}x)^2}{(1-\frac{2}{3}x)^{3/2}} - b.$$

Then equation (2.35) for the extreme α^2 yields

$$\Delta = A/2B; \tag{2.38}$$

at $\lambda = 2\lambda_c$ for example, $\Delta \doteqdot 0{\cdot}0591$ and

$$(\overline{WT})_m|_{\alpha^2=\frac{1}{2}+\Delta} = 1{\cdot}00672(\overline{WT})_m|_{\alpha^2=\frac{1}{2}}.$$

Hence, for this extreme solution, α^2 does change appreciably even in the limited range of λ in which the λ_4-approximation gives an adequate description of the motion. However, the change in heat transport is quite negligible and tells one that this change in α^2 plays an unimportant role in the energetics of our probem.

Though certainly beyond the useful limit of the λ_4-approximation, the 'end point' of the λ_4-curve has been computed as a function of α^2 to complete the extreme $(\overline{WT})_m$-curve labelled (α^2) in figure 2. At this end point

$$\lambda = \tfrac{1}{4}\lambda_2/|\lambda_4| + \lambda_0 \equiv \lambda_{\text{imag}},$$

and

$$(\overline{WT})_m|_{\lambda=\lambda_{\text{imag}}} \doteqdot 4(\lambda_{\text{imag}}-\lambda_0). \tag{2.39}$$

Hence, from (2.35) for the extreme $(\overline{WT})_m$ as a function of α^2, we have, using the values of λ_{imag} and $(\overline{WT})_m$ in (2.39),

$$\partial K/\partial\alpha^2 + 4\,\partial\lambda_0/\partial\alpha^2 = 0. \tag{2.40}$$

It is seen that this condition for maximum heat transport at the end point is identical with the condition for minimum λ_{imag} from (2.39). A plot of λ_{imag} against α^2, using λ_0, λ_2, and λ_4 from (2.34), leads to the (λ, H)-point given in figure 2. The value of α^2 at this minimum λ_{imag} is 27/40.

We have now progressed about as far as is practicable in the ϵ-sequence for the simplest of the solutions to the Rayleigh problem. The two important properties of the non-linear terms which control the amplitude of the steady disturbance have been exhibited. These are the distortion of the disturbance from its initial form and the distortion by the disturbance of the mean temperature field. This latter plays the dominant role in amplitude determination for the free rolls. In the following section we shall establish the relative importance of these two effects for the general rectangle and the hexagon.

3. Analysis for the three-dimensional cellular motion

The greater complexity of the general rectangle and hexagon plan-forms prohibits an analysis as extended as that for the rolls. However, the non-vanishing of the first zero-average advection terms causes even the λ_2-approximation for these plan-forms to exhibit many of the properties found for the roll-cell in the λ_6-approximation.

General rectangle

We start this analysis with the general rectangular cell with two free boundaries. A complete solution to the first-order equations is

$$\left.\begin{aligned} W_0 &= 2\sqrt{2}\cos\pi lx\cos\pi my\sin\pi z, \\ T_0 &= N_0 2\sqrt{2}\cos\pi lx\cos\pi my\sin\pi z, \\ U_0 &= -2\sqrt{2}l\alpha^{-2}\sin\pi lx\cos\pi my\cos\pi z, \\ V_0 &= -2\sqrt{2}m\alpha^{-2}\cos\pi lx\sin\pi my\cos\pi z, \end{aligned}\right\} \tag{3.1}$$

$$\lambda_0 = \pi^4(1+\alpha^2)^3/\alpha^2, \qquad N_0 \equiv \pi^2(1+\alpha^2)^2/\alpha^2, \qquad l^2+m^2 = \alpha^2.$$

From (1.5), (1.8) and (3.1),

$$\left.\begin{aligned} \nabla_1^2 h_{00} &= -8\pi^3 N_0\alpha^{-2}m^2l^2\{\cos 2\pi lx+\cos 2\pi my\}\sin 2\pi z, \\ L_{00} &= 8\pi^3 l^2m^2\alpha^{-4}(1+\alpha^2)[\cos 2\pi lx+\cos 2\pi my]\sin 2\pi z. \end{aligned}\right\} \tag{3.2}$$

Hence $\lambda_1 = 0$ as before and, from (1.20),

$$\mathscr{L}(W_1) = \frac{8\pi^3m^2l^2N_0}{\alpha^2}\left[\left\{1+\frac{4}{\sigma}\frac{(1+l^2)}{(1+\alpha^2)}\right\}\cos 2\pi lx + \right.$$
$$\left. + \left\{1+\frac{4}{\sigma}\frac{(1+m^2)}{(1+\alpha^2)}\right\}\cos 2\pi my\right]\sin 2\pi z. \tag{3.3}$$

The solution of (3.3) is

$$W_1 = (C_{lm}\cos 2\pi lx + C_{ml}\cos 2\pi my)\sin 2\pi z, \tag{3.4}$$

where

$$C_{lm} = -\frac{8\pi^3\alpha^{-2}m^2l^2N_0\{1+4\sigma^{-1}(1+l^2)/(1+\alpha^2)\}}{64\pi^6(1+l^2)^3-4\pi^2l^2\lambda_0}, \qquad C_{lm}(m,l) = C_{ml}.$$

From (1.21) and (3.4)

$$T_1 = (D_{lm}\cos 2\pi lx + D_{ml}\cos 2\pi my)\sin 2\pi z, \tag{3.5}$$

where

$$D_{lm} = \frac{\lambda_0 C_{lm} - 2\pi N_0 m^2\alpha^{-2}}{4\pi^2(1+l^2)}, \qquad D_{lm}(m,l) = D_{ml}.$$

From (1.9) and (1.10) expanded for U_1 and V_1,

$$\partial(M_x)_{00}/\partial y - \partial(M_y)_{00}/\partial x = 0,$$

so that

$$\left.\begin{aligned} U_1 &= -(C_{lm}/l)\sin 2\pi lx\cos 2\pi z, \\ V_1 &= -(C_{ml}/m)\sin 2\pi my\cos 2\pi z. \end{aligned}\right\} \tag{3.6}$$

To determine the first finite amplitude results for the general rectangle we must construct λ_2 from (1.23). Here

$$\lambda_2 = \{\overline{W_0\nabla_1^2 W_0[\overline{W_0 T_0}} - \overline{W_0 T_0}]_m + \overline{W_0\nabla_1^2(h_{01}+h_{10})} +$$
$$+\sigma^{-1}\overline{W_0\nabla^2(L_{01}+L_{10})}\}_m/(\overline{W_0\nabla_1^2 W_0})_m, \tag{3.7}$$

while from (3.1), (3.5) and (3.6) the lengthy forms of h_{ij} and L_{ij} lead to

$$\frac{[\overline{W_0 \nabla_1^2(h_{01}+h_{10})_m}]}{(\overline{W_0 \nabla_1^2 W_0})_m} = -\frac{\pi^2}{2\alpha^2}(m^2 D_{lm} + l^2 D_{ml}) \equiv \tfrac{1}{2}N_0 D(\sigma, l, m) \quad (3.8)$$

and

$$\frac{[\overline{W_0 \nabla^2(L_{01}+L_{10})}]_m}{(\overline{W_0 \nabla_1^2 W_0})_m} = -\frac{N_0 \pi}{2\alpha^2}(m^2 C_{lm} + l^2 C_{ml}) \equiv \tfrac{1}{2}N_0 C(\sigma, l, m).$$

Therefore

$$\lambda_2 = \lambda_2(\sigma, m, l) = \tfrac{1}{2}N_0(1+D+C/\sigma), \quad (3.9)$$

and from (1.18) and (1.19) this first effect of ϵ on heat transport is

$$(\overline{WT})_m = \text{function of } (\sigma, l, m) = \frac{2(\lambda-\lambda_0)}{1+D+C/\sigma}. \quad (3.10)$$

For the special case of a square cell $l^2 = m^2 = \frac{1}{2}\alpha^2$; from (3.4), (3.5) and (3.8)

$$1+D+C/\sigma = 1 + \frac{8x^2}{(1+\alpha^2)(32x^3-1)}\left\{1 + \frac{1}{4x^2\sigma} + \frac{1}{2x\sigma^2}\right\}, \quad (3.11)$$

where $x \equiv (1+\frac{1}{2}\alpha^2)/(1+\alpha^2)$.

For the 'limiting rectangle' $l \to 0$, $m^2 \to \alpha^2$, we have

$$1+D+C/\sigma = 1+\tfrac{1}{2}; \quad (3.12)$$

that is, $W_1 = U_1 = V_1 = 0$, from (3.4) and (3.6), but

$$T_1 = -(N_0/2\pi)\cos 2\pi l x \sin 2\pi z$$

from (3.5).

Figure 4 is a plot of $\frac{1}{2}(\overline{WT})_m/(\lambda-\lambda_0)$ as given by equation (3.10), for values of l/m between these two extremes, with $\alpha^2 = \frac{1}{2}$, and for several values of σ. The dependence of $(\overline{WT})_m$ on α^2 will be discussed shortly.

Finite amplitude effects

Perhaps the most important aspect of these finite amplitude effects is that the heat transport for the limiting rectangle does not coincide with that for the roll. Neither rolls nor long rectangles can satisfy the (distant) lateral boundary conditions in real convection. However, a long rectangle is certainly the more realistic limiting form to compare with squares and hexagons.

A second aspect of this initial amplitude convection is that squares transport more heat than any other rectangle when $\sigma \geqslant 0{\cdot}8$*. Most gases have a value of σ very close to this critical value and may exhibit a far less decided choice of plan-form than liquids. However there is some evidence that these strong effects of σ on heat transport disappear at large Rayleigh numbers (Malkus 1954 a).

As for the roll plan-form, one expects that the value of α^2 can change as the rectangle grows in amplitude. We investigate this effect for the

* When $\sigma \ll 1$ (cf. $\sigma \doteqdot 1/40$ for mercury), the radius of convergence of the $\lambda = \lambda(\epsilon, \sigma^{-1})$ series which we have evolved is very small. In such a case it is better to non-dimensionalize **V** by writing $\mathbf{V} = \delta\mathbf{V}'/d$ and obtain the series $\lambda = \lambda(\epsilon, \sigma)$.

case of the square and, again arbitrarily, seek that value of α^2 which leads to a maximum value of $(\overline{WT})_m$ at $\lambda \geqslant (\lambda_0)_{\min} = \lambda_c$. From (3.10) and (3.11),

$$(\overline{WT})_m = 2(\lambda-\lambda_0)\left\{1+\frac{8x^2}{(1+\alpha^2)(32x^3-1)}\left(1+\frac{1}{4x^3\sigma}+\frac{1}{2x\sigma^2}\right)\right\}^{-1}. \quad (3.13)$$

For values of λ near λ_0, α^2 will be near $\frac{1}{2}$. We write $\Delta \equiv \alpha^2 - \frac{1}{2}$, whence

$$\lambda_0 = \lambda_c(1+4\Delta^2/3+\ldots),$$
$$x = \tfrac{5}{6}(1-4\Delta/15+8\Delta^2/45+\ldots),$$
$$(\overline{WT})_m = \frac{2(\lambda-\lambda_c)}{F}\frac{[1-4\lambda_c\Delta^2/3(\lambda-\lambda_c)+\ldots]}{(1-c\Delta+d\Delta^2+\ldots)}, \quad (3.14)$$

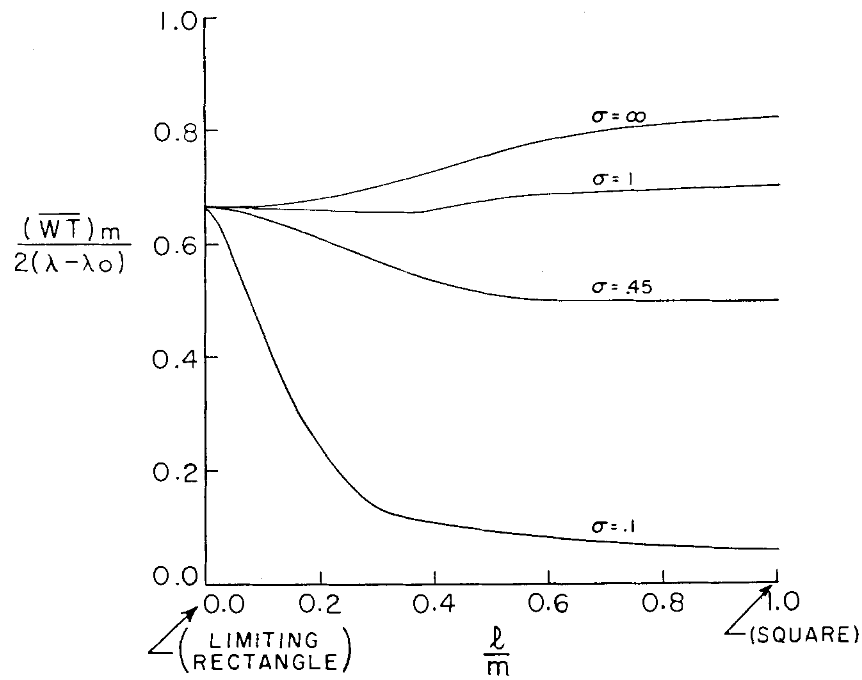

Figure 4. Heat transport for the general rectangle with $\alpha^2 = \frac{1}{2}$ at several values of σ.

where

$$F \equiv 1+\frac{100}{473}\left(1+\frac{9}{25\sigma}+\frac{3}{5\sigma^2}\right),$$
$$c \equiv \frac{100}{473F}(0{\cdot}7124+0{\cdot}0645\sigma^{-1}+0{\cdot}2674\sigma^{-2}),$$
$$d = \frac{100}{473F}(0{\cdot}7800+0{\cdot}0927\sigma^{-1}+0{\cdot}2900\sigma^{-2}).$$

Therefore $\partial(\overline{WT})_m/\partial\Delta = 0$, when the optimum value of Δ is

$$\Delta_{\text{opt}} = \frac{\lambda_c}{\lambda-\lambda_c}\tfrac{1}{2}c\left\{\frac{4}{3}+\frac{\lambda_c}{\lambda-\lambda_c}(d-c^2)\right\}^{-1}$$

and $$[(\overline{WT})_m]_{\text{opt}} = 2(\lambda-\lambda_c)(1+\tfrac{1}{2}c\Delta_{\text{opt}})/F. \tag{3.15}$$

For $\sigma = \infty$, $c \doteqdot 0{\cdot}1245$ and $d \doteqdot 0{\cdot}1362$. When $\lambda = 2\lambda_c$, $\Delta_{\text{opt}} = 0{\cdot}0428$ and

$$[(\overline{WT})_m]_{\text{opt}}/(\overline{WT})_m|_{\alpha^2=\frac{1}{2}} = 1{\cdot}00267.$$

For $\sigma = 1$, $c \doteqdot 0{\cdot}1561$ and $d \doteqdot 0{\cdot}1589$. When $\lambda = 2\lambda_c$, $\Delta_{\text{opt}} = 0{\cdot}0532$ and the above displayed ratio is $1{\cdot}00415$. Hence, for those values of σ for which the square cell transports the most heat, this transport is only negligibly increased by an optimization of α^2. However this optimization causes a 10% increase in α^2 for $\lambda = 2\lambda_0$.

The optimum α^2 equals $\frac{1}{2}$ for the limiting rectangle since from (3.12) λ_2 is independent of α^2.

The range of λ in which these λ_2-approximations for the general rectangular cell are useful is apparently even more restricted than it was for rolls, i.e. $\lambda_c \leqslant \lambda < 2\lambda_c$. This was established by the extremely tedious computation of λ_4 for the square with $\alpha^2 = \frac{1}{2}$. It was found that here $\lambda_4 \doteqdot 0{\cdot}185$, indicating that corrections due to λ_4 become important for $\lambda - \lambda_c > 0{\cdot}5\lambda_c$.

λ_2 for hexagons

Before attempting further interpretation of these results we will investigate the initial finite amplitude convection of the hexagonal plan-form. Past experimental literature gives one the impression that Rayleigh-like convection is realized only as hexagons. However these observations of plan-form were made with rigid bottom and free top boundaries. The one experimental study of plan-form made with symmetric boundary conditions will be discussed shortly.

The complete solution to the first-order problem for a hexagonal plan-form and two free boundaries was first given by Christopherson (1940), and is as follows

$$\left.\begin{aligned}
W_0 &= \frac{2}{\surd 3}\left\{2\cos\frac{2\pi x}{\surd 3L}\cos\frac{2\pi y}{3L} + \cos\frac{4\pi y}{3L}\right\}\sin\pi z \equiv \frac{2}{\surd 3}\phi_1\sin\pi z\\
T_0 &= (2/\surd 3)N_0\phi_1\sin\pi z,\\
U_0 &= -\frac{2}{\alpha}\sin\frac{2\pi x}{\surd 3L}\cos\frac{2\pi y}{3L}\cos\pi z,\\
V_0 &= -\frac{2}{\surd 3\alpha}\left\{\cos\frac{2\pi x}{\surd 3L}\sin\frac{2\pi y}{3L} + \sin\frac{4\pi y}{3L}\right\}\cos\pi z,\\
\lambda_0 &= \pi^4(1+\alpha^2)^3/\alpha^2, \qquad N_0 = \pi^2(1+\alpha^2)^2/\alpha^2,
\end{aligned}\right\} \tag{3.16}$$

where $L = 4/3\alpha$ is the non-dimensional length of one side of the hexagon. Averages over the hexagon can be taken over the region $0 \leqslant x \leqslant \surd 3L/2$ and $0 \leqslant y \leqslant L/2$, which covers one-twelfth of the plan-form and is the smallest symmetric segment.

Possible variations of α^2 from its initial value $\alpha^2 = \frac{1}{2}$ have played such a small role in the initial heat transport of the general rectangle that we simplify the following analysis by choosing $\alpha^2 = \frac{1}{2}$ from the start. Then from (1.5), (1.8) and (3.16),

$$\left.\begin{aligned} \nabla_1^2 h_{00} &= -\tfrac{1}{2}\pi^3 N_0(\phi_1+\phi_2)\sin 2\pi z \\ L_{00} &= (3\pi^2/2)(\phi_1+\phi_2)\sin 2\pi z, \end{aligned}\right\} \tag{3.17}$$

where

$$\phi_2 \equiv 2\cos\frac{2\pi x}{\sqrt{3}L}\cos\frac{2\pi y}{L} + \cos\frac{4\pi x}{\sqrt{3}L}, \qquad \overline{\phi_1\phi_2} = 0, \qquad \nabla_1^2\phi_2 = -(3\pi^2/2)\phi_2.$$

Here, for the first time in our study, h_{00} and L_{00} have a horizontal dependence which is not orthogonal to the plan-form of W_0. This unusual property of the hexagonal solutions will be important in the case of one rigid and one free boundary, where due to the asymmetry the z-dependence of h_{00} and L_{00} and the z-dependence of W_0 are not orthogonal. However in this case of two free boundaries $\lambda_1 = 0$, since $(\sin\pi z \sin 2\pi z)_m = 0$. Therefore, from (1.20),

$$\mathscr{L}(W_1) = \tfrac{1}{2}\pi^3 N_0[(1+3/\sigma)\phi_1 + (1+11/3\sigma)\phi_2]\sin 2\pi z, \tag{3.18}$$

with the solution

$$W_1 = -(9\sqrt{3}/4\pi)(C_1\phi_1 + C_2\phi_2)\sin 2\pi z, \tag{3.19}$$

where

$$C_1 = \frac{1+3/\sigma}{729/4-\lambda c/\pi^4}, \qquad C_2 = \frac{1+11/3\sigma}{1331/4-3\lambda c/\pi^4}.$$

From (1.21) and (3.18), we have

$$T_1 = -\frac{\pi}{2\sqrt{3}}\left[3\left(\frac{729}{4}C_1 - \frac{3}{\sigma}\right)\phi_1 + \frac{9}{11}\left(\frac{1331}{4}C_2 - \frac{11}{3\sigma}\right)\phi_2\right]\sin 2\pi z. \tag{3.20}$$

The comparison of W_1 and T_1 for the hexagon with W_1 and T_1 for the general rectangle assures one that the σ-dependence of these initial finite amplitude distortions is similar. Therefore, a determination of the hexagon λ_2 for $\sigma \to \infty$ will permit an adequate comparison of hexagonal and general rectangle initial heat transports. Consequently the computation of λ_2 from (3.7) is considerably simplified, for we need not compute L_{01} and L_{10}. Then

$$[\overline{W_0\nabla_1^2(h_{01}+h_{10})}]_m = -(\alpha^2\pi^2/N_0)[\overline{T_0(h_{01}+h_{10})}]_m \tag{3.21}$$

from (3.16). Now $(\overline{Th})_m = 0$ in general, and therefore

$$(\overline{T_0 h_{00}})_m = 0, \qquad (\overline{T_1 h_{00} + T_0(h_{01}+h_{10})})_m = 0, \quad \text{etc.}; \tag{3.22}$$

hence

$$[\overline{W_0\nabla_1^2(h_{01}+h_{10})}]_m = (\alpha^2\pi^2/N_0)(\overline{T_1 h_{00}})_m.$$

From (3.22), (3.21) and (3.18),

$$(\overline{T_1 h_{00}})_m = -\frac{27}{8}\pi^4\left\{\frac{729}{4}C_1 - \frac{3}{\sigma} + \frac{1}{11}\left(\frac{1331}{4}C_2 - \frac{11}{3\sigma}\right)\right\} \equiv -\frac{27}{8}\pi^4 C. \tag{3.23}$$

Then from (3.7), (3.23) and (3.16) for $\sigma = \infty$,

$$\lambda_2 = \tfrac{1}{2}N_0 - (\overline{T_1 h_{00}})_m = \tfrac{1}{2}N_0\{1+\tfrac{1}{3}(C)_{\sigma=\infty}\} = (9\pi^2/4)(1\cdot379).$$

The initial finite amplitude heat transport for the hexagon from (1.8), (1.19), and λ_2 above is $(\overline{WT})_m = 1\cdot45(\lambda-\lambda_0)$. In figure 4 we see that this heat transport is smaller than that due to the square and larger than that due to the limiting rectangle.

Mixed horizontal plan-form

There is yet another class of solutions to the first-order equations which may grow to finite amplitude, viz. the class of all possible linear combinations of the individual plan-forms. We will consider only the single case of a square (S) and a general rectangle (G). The first-order solution will be

$$W_0 = \surd 2(\phi_S + r\phi_G)(1+r^2)^{-1/2}\sin\pi z \equiv \surd 2\phi\sin\pi z, \tag{3.24}$$

where

$$\phi_S = 2\cos(\pi\alpha x/\surd 2)\cos(\pi\alpha y/\surd 2), \qquad \overline{\phi^2} = 1,$$
$$\phi_G = 2\cos\pi l x\cos\pi m y, \qquad l^2+m^2 = \alpha^2,$$

and $0 \leqslant r \leqslant \infty$ is a free parameter determining the fractional contribution of ϕ_G to the total plan-form. In the case $\alpha^2 = \frac{1}{2}$, $\sigma = \infty$, we can quickly determine λ_2 by the method used for the hexagon (see equation (3.23)). From (3.24),

$$h_{00} = \pi N_0(1+r^2)^{-1}\{h_{00}(S) + r^2 h_{00}(G) + \\ + 2r(1\mp l\mp m)\cos(\tfrac{1}{2}\pm l)\pi x\cos(\tfrac{1}{2}\pm m)\pi y\sin 2\pi z\}. \tag{3.25}$$

Hence

$$W_1 = (1+r^2)^{-1}\{W_1(S) + r^2W_1(G) + \\ + (2rB/\pi^3)(4+A)^{-2}\cos(\tfrac{1}{2}\pm l)\pi x\cos(\tfrac{1}{2}\pm m)\pi y\sin 2\pi z\} \tag{3.26}$$

and

$$T_1 = (1+r^2)^{-1}\{T_1(S)+r^2T_1(G) - (2rB/\pi)\cos(\tfrac{1}{2}\pm l)\pi x\cos(\tfrac{1}{2}\pm m)\pi y\sin 2\pi z\}$$

where

$$A = (\tfrac{1}{2}\pm l)^2 + (\tfrac{1}{2}\pm m)^2, \quad B = N_0(1\mp l\mp m)(4+A)^2/[(4+A)^3 - \lambda c A/\pi^4].$$

Then

$$(\overline{T_1 h_{00}})_m = (1+r^2)^{-2}\{(\overline{T_1 h_{00}})_m(S) + r^4(\overline{T_1 h_{00}})_m(G) - \tfrac{1}{2}r^2 N_0 B(1\mp l\mp m)\}. \tag{3.27}$$

Finally, from (3.24), (3.9) and (3.27)

$$\lambda_2 = (1+r^2)^{-2}\{\lambda_2(S) + 2r^2\lambda_2(I) + r^4\lambda_2(G)\}, \tag{3.28}$$

where

$$\lambda_2(I) \equiv \tfrac{1}{2}N_0 + \tfrac{1}{4}B(1\mp l\mp m).$$

Hence λ_2 has an extremum with respect to r at

$$(r^2)_{\text{opt}} = [\lambda_2(S)-\lambda_2(I)]/[\lambda_2(G)-\lambda_2(I)],$$

where

$$(\lambda_2)_{\text{opt}} = [\lambda_2(S)\lambda_2(G) - \lambda_2^2(I)]/[\lambda_2(S)+\lambda_2(G)-2\lambda_2(I)]. \tag{3.29}$$

This extreme is a maximum if $2\lambda_2(G)\lambda_2(I) \geqslant \lambda_2^2(I)+\lambda_2^2(G)$. Comparison of (3.9) for $\lambda_2(G)$ with $\sigma = \infty$, $\alpha^2 = \frac{1}{2}$, and (3.28) for $\lambda_2(I)$ establishes that this inequality is true for all values of $l = (\frac{1}{2} - m^2)^{1/2}$. We recall that an increase in λ_2 decreases the heat transport. Therefore, as one might have anticipated, this mixture of first-order plan-forms reduces the initial finite amplitude heat transport.

Before a formal attempt is made to determine the physically realized solution from the vast degenerate set we have constructed, a relevant experimental study will be discussed. In 1935 Schmidt & Milverton made optical observations of the spacing of convecting cells between two rigid boundaries. Shining a collimated beam of light horizontally through water ($\sigma = 8$) they detected a spatial oscillation in intensity due to the density variations in each cell. The patterns obtained by rotating the beam through 180° in the horizontal contain one maximum in the spacing of the light fringes in the case of limiting rectangles, two maxima for squares, and three maxima for hexagons. At these maxima, from the initial $\alpha^2 = 3{\cdot}13/\pi$ (appropriate to the case of two rigid boundaries) the ratio of horizontal extent of each cell to the vertical spacing of the boundaries will be 2 for squares and limiting rectangles and 1·8 for hexagons. Schmidt & Milverton do not record any rotation of their beam to determine uniquely the plan-form. However, the average ratio of horizontal fringe spacing to cell height in their experiment was 2·1. They believed that they were observing squares. We can tentatively conclude that these cells were either squares or limiting rectangles, but not hexagons.

The qualitative aspects of our heat transport computations for various plan-forms will most probably be preserved in going from the symmetric case of two free boundaries to the symmetric case of two rigid boundaries. For two free boundaries we found that the square transports more heat, the limiting rectangle less heat than the hexagon. Hence one anticipates that heat transport is a selective factor in establishing the observed flow from the many solutions.

4. A 'RELATIVE STABILITY' CRITERION

When seeking a realizable steady-state solution to the equations of motion one must verify not only that the solution formally satisfies the time-independent equations but also that it is stable against all infinitesimal disturbances which satisfy the boundary conditions. If it is not stable then surely there is at least one other solution to the equations. This other solution may not be a steady one in the local sense, but for fixed boundary conditions is must be statistically steady; that is, integrals of moments of this solution over the entire field will be independent of time.

The general problem of the stability of a known solution $\mathbf{v}$ and $\mathbf{T}$ can be immensely complicated. For example, years of labour have been devoted to simple two-dimensional shear flow (the Orr–Sommerfeld problem) and no exact solution has yet been found. Hence it is improbable that we could formally determine the stability of finite amplitude solutions even if we knew them exactly.

Fortunately the answer to a simpler question will suffice here. We will assume that we know the complete set of steady and statistically steady solutions to the equations of motion. Then any experimentally realized solution is contained in this set. We ask, are any of these solutions stable against those infinitesimal disturbances that have the form of the other solutions? If there is an affirmative answer for just one of these solutions, we will have determined the realized solution. If there are several solutions which are 'relatively stable', then either these solutions can be realized separately (i.e. they are metastable) or a broader class of infinitesimal disturbances must be considered to remove the remaining indeterminancy.

In formulating such a question we have hoped that a detailed knowledge of the individual solutions and their interactions would not be required to answer it. Indeed we seek some simple integral property of the solutions which will single out the stable one.

From (1.1), (1.2) and (1.3) in dimensional form, the equations of an arbitrary disturbance $\mathbf{v}'$, T' on the field $\mathbf{v}$, $\mathbf{T}$ are

$$(\partial/\partial t - \kappa\nabla^2)T' = -\mathbf{v}\,.\,\nabla T' - \mathbf{v}'\,.\,\nabla\mathbf{T} - \mathbf{v}'\,.\,\nabla T', \tag{4.1}$$

$$(\partial/\partial t - \nu\nabla^2)\mathbf{v}' = -\mathbf{v}\,.\,\nabla\mathbf{v}' - \mathbf{v}'\,.\,\nabla\mathbf{v} - \mathbf{v}'\,.\,\nabla\mathbf{v}' - \nabla\widetilde{P}'/\rho_m + \gamma T'\mathbf{k}, \tag{4.2}$$

and

$$\nabla\,.\,\mathbf{v}' = 0. \tag{4.3}$$

These are the general equations which determine the stability of $\mathbf{v}$, $\mathbf{T}$. We first construct the disturbance power integrals (see (1.6) and (1.7)),

$$(\gamma/2\beta_m)\partial(\overline{T'^2})_m/\partial t = (\gamma/\beta_m)[\kappa(\overline{T'\nabla^2 T'})_m + (\overline{W'T'\beta})_m - (\overline{T'\mathbf{v}'\,.\,\nabla T})_m] \tag{4.4}$$

and

$$\tfrac{1}{2}\,\partial(\overline{\mathbf{v}'\,.\,\mathbf{v}'})_m/\partial t = \nu(\overline{\mathbf{v}'\,.\,\nabla^2\mathbf{v}'})_m + \gamma(\overline{W'T'})_m - (\overline{\mathbf{v}'\,.\,\mathbf{v}'\,.\,\nabla\mathbf{v}})_m, \tag{4.5}$$

where $\mathbf{T}$ has been replaced by $\overline{\mathbf{T}} + T$ and $\beta = -\partial\overline{\mathbf{T}}/\partial z$. If $\mathbf{v}$ and $\mathbf{T}$ are stable to the disturbance $\mathbf{v}'$, T', the right-hand sides of (4.4) and (4.5) will be negative. One may interpret (4.4) as the equation for the balance of entropy* associated with the disturbance T'. Equation (4.5) is the equation for the balance of kinetic energy of the disturbance $\mathbf{v}'$. The first term on the right-hand side of (4.4) is proportional to the loss of entropy by thermal diffusion; the second is proportional to the gain of entropy through interaction with the initial mean temperature gradient β; and the third is proportional to the gain or loss of entropy through interaction with the initial field of temperature fluctuation. In (4.5) the first term is the loss of kinetic energy of the fluctuation $\mathbf{v}'$ by viscosity, the second is the production of kinetic energy by convection and the last is the gain or loss of kinetic energy through interaction with the initial velocity field.

We will now restrict the class of disturbances to be considered to those which have the form of steady or statistically steady solutions to the basic

* In this irreversible process of steady convection we regard 'entropy' as proportional to and as a shorthand for the mean square of the local fluctuations of internal energy. We prescribe no (thermodynamic) role for this 'entropy' which cannot be deduced from the basic disturbance equation (4.1).

equations; that is,

$$\mathbf{v}' = A\mathbf{v}_1, \qquad T' = BT_1 \tag{4.6}$$

where A and B are arbitrary amplitudes and $\mathbf{v}_1$, T_1 are correct solutions. Then from (4.4), (4.5) and (4.6),

$$\frac{\partial B}{\partial t} = -I_1 B + I_2 A, \qquad \frac{\partial A}{\partial t} = -I_3 A + I_4 B, \tag{4.7}$$

where

$$I_1 \equiv -\kappa(\overline{T_1 \nabla^2 T_1})_m/(\overline{T_1^2})_m, \qquad I_2 \equiv [(\overline{W_1 T_1 \beta})_m - f_T]/(\overline{T_1^2})_m,$$

$$f_T \equiv (\overline{\mathbf{v}_1 T_1 . \nabla T})_m, \qquad I_3 \equiv -[\nu(\overline{\mathbf{v}_1 . \nabla^2 \mathbf{v}_1})_m + f_v]/(\overline{\mathbf{v}_1 . \mathbf{v}_1})_m,$$

$$f_v \equiv (\overline{\mathbf{v}_1 . \mathbf{v}_1 . \nabla \mathbf{v}})_m, \qquad I_4 \equiv \gamma(\overline{W_1 T_1})_m/(\overline{\mathbf{v}_1 . \mathbf{v}_1})_m.$$

The solution of (4.7) will decay with time if

$$[\tfrac{1}{4}(I_1 + I_3)^2 - (I_1 I_3 - I_2 I_4)]^{1/2} - \tfrac{1}{2}(I_1 + I_3) < 0. \tag{4.8}$$

But $(I_1 + I_3)$ contains the two large positive-definite diffusive dissipation terms and is invariably positive. Therefore decay will occur if

$$I_1 I_3 > I_2 I_4. \tag{4.9}$$

Since $\mathbf{v}_1$, T_1 are solutions,

$$\left.\begin{aligned} \kappa(\overline{T_1 \nabla^2 T_1})_m + (\overline{W_1 T_1 \beta_1})_m &= 0, \\ \nu(\overline{\mathbf{v}_1 . \nabla^2 \mathbf{v}_1})_m + \gamma(\overline{W_1 T_1})_m &= 0. \end{aligned}\right\} \tag{4.10}$$

With the use of (4.10) the stability criterion (4.9) becomes

$$[\overline{W_1 T_1}(\beta_1 - \beta)]_m + f_T + f_v(\overline{W_1 T_1 \beta_1})_m/\gamma(\overline{W_1 T_1})_m > 0. \tag{4.11}$$

The basic equations for $\mathbf{v}_1$, T_1 allow the triple product integrals f_v and f_T to be expressed as double product integrals, that is,

$$\left.\begin{aligned} f_T = (\overline{\mathbf{v}_1 . T_1 \nabla T})_m &\equiv -(\overline{T\mathbf{v}_1 . \nabla T_1})_m \\ &= [\overline{T(\partial T_1/\partial t - \kappa\nabla^2 T_1 - W_1 \beta_1)}]_m, \\ f_v = (\overline{\mathbf{v} . \partial \mathbf{v}_1/\partial t} &- \nu\overline{\mathbf{v} . \nabla^2 \mathbf{v}_1} - \gamma \overline{WT_1})_m. \end{aligned}\right\} \tag{4.12}$$

All the double product terms of (4.12) vanish in horizontal integration if the field $\mathbf{v}, T$ is developed from a fundamental periodic function orthogonal to the fundamental function of $\mathbf{v}_1$, T_1. This is the case for all different solutions which can be generated by the iterative method of the previous sections (for $\lambda < 2^4\lambda_0$). Hence in our problem f_v and f_T vanish. To determine from (4.11) which of the many solutions is stable we must find that solution for which $[\overline{W_i T_i}(\beta_i - \beta)]_m > 0$, where the index i ranges over all solutions but the one considered. A physical interpretation of this conclusion is that the stable solution will produce more entropy per unit time from the mean temperature field of any other solution than it does from its own mean field.

The stability criterion can be further simplified and given another physical interpretation. Since $H = \kappa\beta + \overline{WT}$ for all solutions, then

$$[\overline{W_i T_i}(\beta_i - \beta)]_m = \kappa(\beta\beta_i - \beta_i^2)_m > 0. \tag{4.13}$$

Condition (4.13) is not satisfied if $(\beta_i^2)_m \geqslant (\beta^2)_m$, for on applying Schwarz's inequality we have $(\beta_i^2)_m^2 \geqslant (\beta^2)_m(\beta_i^2)_m \geqslant (\beta\beta_i)_m^2$. Therefore a necessary condition for stability is that

$$(\beta^2)_m > (\beta_i^2)_m, \tag{4.14}$$

that is, the stable solution has a greater mean-square gradient than any other solution. If there is only one such solution, the criterion (4.14) uniquely resolves the formal degeneracy. If several solutions have the same maximum $(\beta^2)_m$, they are either metastable or a larger class of infinitesimal disturbances must be considered to decide which is realized.

Yet another physical meaning can be given to the maximum $(\beta^2)_m$ criterion. From (4.10),

$$\frac{\beta^2}{\beta_m^2} - 1 = \left(\frac{\overline{WT}}{\kappa\beta_m}\right)_m - \left(\frac{\beta\overline{WT}}{\kappa\beta_m^2}\right)_m = \frac{1}{\gamma\kappa\beta_m}\left\{\nu(-\overline{\mathbf{v}\cdot\nabla^2\mathbf{v}})_m - \frac{\gamma\kappa}{\beta_m}(-\overline{T\nabla^2 T})_m\right\}, \tag{4.15}$$

which states that for the stable solution the rate of dissipation of kinetic energy minus a quantity proportional to the rate of increase of entropy by thermal diffusion is a maximum. Equation (4.15) may also be written

$$(\beta^2/\beta_m^2)_m - 1 = N_2 F,$$

where $$N \equiv (\overline{WT})_m/\kappa\beta_m, \qquad F \equiv (\overline{WT^2})_m/(\overline{WT})_m^2 - 1, \tag{4.16}$$

The quantity N depends only upon the amplitude of the convective heat transport. The quantity F depends only upon the z form of the heat transport. But all the solutions for λ_2 found in § 3 have identical z forms for $\overline{WT}$. Therefore the only stable solution found in § 3 is the one of maximum convective heat transport. For $\sigma \geqslant 1$ this solution is the square.

A summary of our conclusions on relative stability is as follows.

(*a*) Maximum heat transport is the stability criterion for all symmetric λ_2 solutions.

(*b*) Maximum $(\beta^2)_m$ is the stability criterion for all solutions generated by the iteration of orthogonal linear forms (which includes all in this paper).

(*c*) Equation (4.11) is the most general stability criterion for vertical convection.

It is of some interest to relate the total potential energy of the convecting fluid to these stability conditions. In general the potential energy is $V = g\rho z$. Hence

$$(\overline{V})_m = \frac{g}{d}\int_{d/2}^{d/2} z\bar{\rho}\, dz = \frac{g}{d}\int_{d/2}^{d/2} z \int_0^z \frac{\partial\bar{\rho}}{\partial z'}\, dz'\, dz, \tag{4.17}$$

where z is now measured from the mid-point of a symmetric convective field. From our density–temperature relation

$$\partial\bar{\rho}/\partial z' = \rho_0\alpha\beta, \tag{4.18}$$

so that

$$(\overline{V})_m = \frac{\gamma\rho_0 d^2\beta_m}{12}\left[1 - \left\{\frac{6}{d^3}\int_{d/2}^{d/2} z^2\frac{\beta}{\beta_m}\, dz - \frac{1}{2}\right\}\right]. \tag{4.19}$$

The expression within curly brackets is zero when $\beta = \beta_m$ and approaches unity as β approaches zero in all but the boundary regions. In steady convection

$$\beta/\beta_m = 1 + [(\overline{WT})_m - \overline{WT}]/\kappa\beta_m,$$

so that

$$(\overline{V})_m = \frac{\gamma\rho_0 d^3 \beta_m}{12}\left[1 - \frac{(\overline{WT})_m}{2\kappa\beta_m}\left\{1 - \frac{3}{d}\int_{-d/2}^{d/2} \frac{z^2}{d^2}\frac{\overline{WT}}{(\overline{WT})_m}\,dz\right\}\right]. \quad (4.20)$$

In comparing two solutions of identical form, i.e. with identical values of $\overline{WT}/(\overline{WT})_m$, equation (4.20) asserts that the one of maximum heat transport will produce a minimum $(\overline{V})_m$. This conclusion is in keeping with the stability condition for λ_2 deduced from (4.16). However, the general condition for relative stability is not the requirement that the total potential energy be a minimum though such a relation might have been supposed from considerations based on equilibrium mechanics.

5. Extension to rigid surface boundary conditions

In this final section we will discuss the determination of finite amplitude effects for two rigid boundaries and for one rigid and one free boundary. We can hardly expect the rigid boundaries to modify the qualitative properties of the free boundary convection, for the important physical processes and the symmetry remain unaltered. We can expect quantitative changes due to the additional constraint on motion near the boundary. However, the asymmetry introduced by the rigid–free boundaries can modify the dependence of initial growth on λ and, as experiments bear out, can lead to hexagons as the preferred plan-form. We shall find that the significant qualitative effects of these boundary conditions can be deduced without the final lengthy numerical computation for the λ_i.

The complexity of these computations is perhaps the best measure of the limitations of the finite amplitude method studied in this paper. In conclusion we will turn from them to discuss alternative methods for determining finite amplitude effects.

Rigid boundary analysis

Proceeding just as in the simpler case, one writes the first-order solution for two rigid boundaries, as given by equations (2.9), (2.10) and (1.21), as

$$\left.\begin{aligned} W_0 &= \sum_1^3 A_i \cosh 2\mu_i z\,\phi(x,y) \equiv W\phi, \\ T_0 &= a^{-2}\sum_1^3 (4\mu_i^2 - a^2) A_i \cosh 2\mu_i z\,\phi(x,y) \equiv t\phi, \end{aligned}\right\} \quad (5.1)$$

where z is now measured from the mid-point, $-\frac{1}{2} \leqslant z \leqslant \frac{1}{2}$, $a^2 = \alpha^2\pi^2$, A_i/A_1 and μ_i are the complex numbers determined by Pellew & Southwell, A_1 is chosen to normalize W_0, and ϕ is the normalized plan-form. In general

$$-\nabla_1^2 h_{00} = 2[-(\nabla_1^2\phi^2) W\,\partial t/\partial z + (-\nabla_1^2\psi) t\,\partial W/\partial z], \quad (5.2)$$

$$L_{00} = 8\left\{(\phi^2-\psi^2)\left(\frac{\partial W}{\partial z}\frac{\partial^2 W}{\partial z^2} - W\frac{\partial^3 W}{\partial z^3}\right)+\Gamma^2\left(\frac{\partial W}{\partial z}\frac{\partial^2 W}{\partial z^2} - a^2 W\frac{\partial W}{\partial z}\right)\right\}, \tag{5.3}$$

where

$$\psi \equiv a^{-2}\left[\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2\right], \qquad \Gamma^2 \equiv 4a^{-4}\left[\left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2 - \frac{\partial^2\phi}{\partial x^2}\frac{\partial^2\phi}{\partial y^2}\right].$$

Therefore

$$-\nabla_1^2 h_{00} = a^{-2}\sum_{i=1}^{3}\sum_{j=-3}^{3} A_i A_j \mu_i[(-\nabla_1^2\phi)(4\mu_i^2-a^2)^2+(-\nabla_1^2\psi)(4\mu_j^2-a^2)^2]\times \times\sinh 2(\mu_i+\mu_j)z,$$

and

$$L_{00} = 4\sum_{i=1}^{3}\sum_{j=-3}^{3} A_i A_j \mu_i[(\phi^2-\psi^2)(\mu_j^2-\mu_i^2)+\Gamma^2(4\mu_j^2-a^2)]\sinh 2(\mu_i+\mu_j)z, \tag{5.4}$$

where $\mu_j \equiv -\mu_{-j}$, $A_j \equiv A_{-j}$, $A_0 \equiv 0$. Hence, due to symmetry alone, $\lambda_1 = 0$, because $[\cosh 2\mu_i z \sinh 2(\mu_i+\mu_j)z]_m = 0$. We let

$$W_1 = \tfrac{1}{2}\sum_{ij}\sum_{k} P_{kij}\sinh 2(\mu_i+\mu_j)z, \tag{5.5}$$

where $-\nabla_1^2 P_{kij} = a_k^2 P_{kij}$. Then from (1.20) for W_1,

$$\begin{aligned}[\{4(\mu_i+\mu_j)^2-a_k^2\}^3+a_k^2\lambda_0]P_{kij} &= 2A_iA_j\alpha^{-2}\mu_i\{(-\nabla_1^2\phi^2)(4\mu_i^2-a^2)^2+(-\nabla_1^2\psi^2)(4\mu_j^2-a^2)^2+\\ &+4a^2\sigma^{-1}[-\nabla_1^2(\phi^2-\psi^2)-4(\mu_i+\mu_j)^2(\phi^2-\psi^2)](\mu_i^2-\mu_j^2)+\\ &+a^2\sigma^{-1}[-\nabla_1^2\Gamma^2-4(\mu_i+\mu_j)^2\Gamma^2](4\mu_i^2-a^2)\}_k. \end{aligned} \tag{5.6}$$

As we saw in § 3, k sums over the three orthogonal functions $\cos 2\pi lx$, $\cos 2\pi my$ and $\cos 2\pi lx \cos 2\pi my$ generated in h_{00} and L_{00} by the general rectangle, and over the two orthogonal functions ϕ_1 and ϕ_2 of (3.18) for the hexagon. The T_1 determined by this W_1 is written

$$T_1 = \tfrac{1}{2}\sum_{ij}\sum_{k} G_{kij}\sinh 2(\mu_i+\mu_j)z \tag{5.7}$$

where

$$a_k^2 G_{kij} = P_{kij}\{4(\mu_i+\mu_j)^2-a_k^2\}^2+2\sigma^{-1}A_iA_j\mu_i\{4(\phi^2-\psi^2)(\mu_i^2-\mu_j^2)+ + \Gamma^2(4\mu_i^2-a^2)\}_k.$$

Recall that

$$\lambda_2 = (\overline{W_0\nabla_1^2 W_0})_m^{-1}\{[\overline{W_0 T_0}-(\overline{W_0 T_0})_m]\overline{W_0\nabla_1^2 W_0}+\overline{W_0\nabla_1^2(h_{01}+h_{10})}+ +\sigma^{-1}\overline{W_0\nabla^2(L_{01}+L_{10})}\}_m, \tag{5.8}$$

showing that the first finite amplitude results depend upon both the initial disturbance and the first distortion of this disturbance W_1, T_1, U_1, and V_1. The only computationally difficult term in λ_2 due to the (W_0, T_0)-field alone is

$$(\overline{W_0 T_0}\,\overline{W_0\nabla_1^2 W_0})_m = -\frac{1}{8}\sum_{\substack{i,r=1\\ i,s=-3}}^{3} A_iA_jA_rA_s(4\mu_3^2-a^2)^2\times \times\left[\frac{\sinh(\mu_i+\mu_j+\mu_r+\mu_s)}{\mu_i+\mu_j+\mu_r+\mu_s}+\frac{\sinh(\mu_i+\mu_j-\mu_r-\mu_s)}{\mu_i+\mu_j-\mu_r-\mu_s}\right] \tag{5.9}$$

which is the sum of 324 complex numbers. But the last two terms in λ_2 caused by the distortion introduce 1944 additional complex numbers. The task of computing the numbers involved in (5.9) for several plan-forms can hardly be justified at this time. However, these numbers determine our only finite amplitude result rigorously applicable to a realizable experiment and could be used to check the physical validity of the formalism.

Analysis for one rigid–one free boundary

The case of one rigid and one free boundary leads to the first-order solution

$$W_0 = \sum_1^3 A_i \sinh 2\mu_i z, \tag{5.10}$$

where $0 \leqslant z \leqslant \frac{1}{2}$ and A_i/A_1 and μ_i are known complex numbers as before. Comparing (5.11) and (5.1), we see that $\nabla_1^2 h_{00} + \sigma^{-1}\nabla^2 L_{00}$ must be identical in form to (5.4), with the change that the index i runs from -3 to 3 while the index j runs from 1 to 3. Hence

$$\begin{aligned}(W_0 \nabla_1^2 W_0)_m \lambda_1 &= [\overline{W_0(\nabla_1^2 h_{00} + \sigma^{-1}\nabla^2 L_{00})}]_m \\ &= -a^{-2} \sum_{\substack{r,j=1 \\ i=-3}}^{3} A_r A_i A_j \mu_i \Big[\sum_k \overline{P'_{kij}\phi}\Big] \times \\ &\qquad \times [\sinh 2\mu_r z \sinh 2(\mu_i + \mu_j) z]_m \end{aligned} \tag{5.11}$$

where

$$\begin{aligned} P'_{kij} = \{&[(4\mu_i^2 - a^2)^2(-\nabla_1^2\phi^2) + (4\mu_j^2 - a^2)^2(-\nabla_1^2\psi^2)] - \\ &- 4a^2\sigma^{-1}(\mu_j^2 - \mu_i^2)[\nabla_1^2(\phi^2 - \psi^2) + 4(\mu_i + \mu_j)^2(\phi^2 - \psi^2)] - \\ &- a^2\sigma^{-1}(4\mu_i^2 - a^2)[\nabla_1^2 \Gamma^2 + 4(\mu_i + \mu_j)^2 \Gamma^2]\}_k. \end{aligned} \tag{5.12}$$

In contrast with previous computations of λ_1, the z-function average $[\sinh 2\mu_r z \sinh 2(\mu_i + \mu_j) z]_m$ is not zero, while, as was noted in § 3, $\overline{P'_{kij}\phi}$ will not vanish for hexagons. Therefore a finite λ_1 exists for hexagons with one rigid and one free boundary. Similarly $(\overline{W_0 T_1} + \overline{W_1 T_0})_m$ will not vanish and must be included in the heat transport. We will not proceed further with this particular evaluation of λ_1 and $(\overline{W_0 T_1} + \overline{W_1 T_0})_m$ but will enquire into the qualitative consequences of their existence.

First-order theory is indeterminate with respect to the sign of the motion in the centre of hexagons. It could be either up or down. In contrast the sign of the amplitude of a rectangle is not a true degeneracy, since a change of sign leads to no observable change in the field of motion. The conclusion that λ_1 is not zero uniquely removes this degeneracy in the hexagon, for λ_1 determines the sign of the amplitude ϵ. From (1.19), $\epsilon = (\lambda - \lambda_0)/\lambda_1$ for the initial convection, and hence if λ_1 is positive ϵ is positive and the motion is up in the hexagon centre, whereas if λ_1 is negative ϵ is negative and the motion is down. We have estimated λ_1 by graphical methods for $\sigma = 8$ and find that $\lambda_1 \doteqdot 9{\cdot}5$. Hence the motion is up in the centre of hexagons as is observed.

One must now determine under what conditions the finite amplitude hexagon is preferred to the square or rectangle in the case of one rigid and one free boundary. To the λ_2-approximation,

$$\epsilon^2\lambda_2 + \epsilon\lambda_1 - (\lambda - \lambda_0) = 0;$$

hence

$$\epsilon = (\lambda_1/2\lambda_2)[\{1 + 4\lambda_2(\lambda - \lambda_0)/\lambda_1\}^{1/2} - 1], \tag{5.13}$$

where the second root is discarded since it does not vanish when $\lambda = \lambda_0$. The corresponding heat transport from (5.13) is

$$(\overline{WT})_m \doteqdot [(\lambda - \lambda_0)/\lambda_2 - \epsilon\lambda_1/\lambda_2][(\overline{W_0 T_0})_m + \epsilon(\overline{W_0 T_1} + \overline{W_1 T_0})_m], \tag{5.14}$$

where we have kept all terms in $(\overline{WT})_m$ necessary in the determination of λ_2. Equation (5.14) asserts that the heat transport will be less than that due to λ_2 alone for values of $(\lambda - \lambda_0)$ close to zero. However if $\lambda_1(\overline{W_0 T_1} + \overline{W_1 T_0})_m^{-1}$ ($= Q$ say) is positive the heat transport will exceed the value appropriate to λ_2 when

$$\lambda - \lambda_0 \geqslant |\lambda_1| (\overline{W_0 T_0})_m^{1/2} (Q/\lambda_2)^{1/2} + (\overline{W_0 T_0})_m Q. \tag{5.15}$$

Since the value of λ_2 for hexagons and the value of λ_2 for squares will be comparable (certainly within a factor of two), then hexagons can become the preferred disturbance if $\lambda_1(\overline{W_0 T_1} + \overline{W_1 T_0})_m^{-1}$ is positive and if the critical value of $(\lambda - \lambda_0)$ as given by (5.15) is small enough so that (5.14) is still valid. We have not yet determined the magnitude of $(\overline{W_0 T_1} + \overline{W_1 T_0})_m$ but a graphical method of integration assures us that its sign is positive. Hence we conclude that hexagons will not appear as the initial instability with one rigid and one free boundary, but will appear for some finite $(\lambda - \lambda_0)$ with positive vertical velocity in their centres. There is some experimental evidence to support this conclusion. Benard (1901) describes the initial appearance of convection due to cooling a fluid from above as "a disordered cellular regime" which becomes a steady field of hexagons after the cooling has continued a short time.

Amplitude determination by integral techniques

The lengthy task of computation needed to determine finite amplitude effects for realistic boundary conditions encouraged us to consider less exact methods. Stuart (1958) describes the use of power integrals to determine the amplitude of any disturbance whose form is assumed to be a good approximation to a correct solution. Stuart's study was concerned with finite amplitude processes in shearing flow, but he has discussed with us the application of this method to thermal convection. We will briefly outline our impressions of the virtues and limitations of the thermal power integrals as tools for finite amplitude study. In particular, we will compute the heat transport for the case of two rigid boundaries which can be compared with the experimental results.

In the steady state the kinetic power integral (1.7) is a homogeneous relation between the amplitudes and forms of $\mathbf{v}$ and T. If $\mathbf{v} = A\mathbf{v}'$ and

$T = BT'$, where T' and the W' component of $\mathbf{v}'$ are normalized, then (1.7) in non-dimensional form may be written

$$B/A = \nu(-\overline{\mathbf{v}'.\nabla^2\mathbf{v}'})_m/\gamma(\overline{W'T'})_m. \tag{5.16}$$

On the other hand the thermal power integral equation (1.6) is an inhomogeneous relation since β/β_m from (1.13) also contains the amplitude. Equations (1.6) and (1.13) may be written

$$AB = (\lambda - \lambda_S)/S, \tag{5.17}$$

where

$$S = (\overline{W'T'})_m \left\{ \frac{(\overline{W'T'^2})_m}{(\overline{W'T'})_m^2} - 1 \right\}, \qquad \lambda_S = \frac{B}{A} \frac{(-\overline{T'\nabla^2 T'})_m}{(\overline{W'T'})_m}.$$

Equation (5.17) for the convective heat transport is like the observed linear relation between heat transport and λ, since S and λ_S will be constants for any particular form of $\mathbf{v}'$ and T'. If we use the form of the initial Rayleigh instability of the case of two free boundaries for $\mathbf{v}'$ and T', then from (2.7),

$$(\overline{W'T'})_m = 1, \qquad S = \tfrac{1}{2}, \qquad \lambda_S = \lambda_0,$$

and

$$AB = 2(\lambda - \lambda_0), \tag{5.18}$$

independent of the plan-form of the initial disturbance. Equation (5.18) is identical with our results for λ_2 in the case of roll cells with free boundaries. This is understandable, for the free roll is the one case in which U_1, V_1, W_1, T_1 distortions and σ effects did not influence initial growth. Since distortions and σ effects uniformly tend to reduce amplitude we may hope that this use of the power integral sets an upper limit on heat transport.

Other inhomogeneous integral relations may be deduced from the heat equation which lead to different but lower values of the heat transport at the same λ and for the same choice of $\mathbf{v}'$, T'. For example we may write the non-dimensional heat equation from equations (1.5) and (1.13) as

$$\lambda W = h + [\overline{WT} - (\overline{WT})_m] W - \nabla^2 T \equiv L. \tag{5.19}$$

Then squaring both sides and averaging we obtain the inhomogeneous relation

$$\lambda^2 = (\overline{L^2})_m / (\overline{W^2})_m, \tag{5.20}$$

which must be satisfied by any correct solution to the problem. If we again use the form of the initial Rayleigh instability for the case of two free boundaries for $\mathbf{v}'$ and T', but for simplicity use the roll plan-form (since (5.20) depends upon plan-form), then

$$AB = (2\lambda^2 - \lambda_0^2)^{1/2} - \lambda_0. \tag{5.21}$$

Thus we have derived a different law for heat transport from that given by the power integral. In general, equation (5.21) yields a lower heat transport for given λ than does (5.18). Only for $\lambda \doteqdot \lambda_0$ do the two formulae agree since it is only at infinitesimal AB that this choice of $\mathbf{v}'$ and T' is a correct solution. In spite of this difficulty, the power-integral heat transports seem to bear too remarkable a resemblance to the observations to be fortuitous, as we will now show by a computation of the Rayleigh function for rigid boundaries paralleling (5.18).

Pellew & Southwell have proposed an approximate solution to the first-order characteristic functions (for rigid boundaries) which is easier to work with than the correct solution,

$$W' = N^{1/2}[\sin n\pi z + P\cosh \pi(z-\tfrac{1}{2}) + Q\pi(z-\tfrac{1}{2})\sinh \pi(z-\tfrac{1}{2})]\phi, \quad (5.22)$$

and

$$T' = \sqrt{2}\phi \sin \pi z,$$

where ϕ is the normalized plan-form, $N^{1/2}$ is a normalizing constant, $P = 0{\cdot}4921$ and $Q = 0{\cdot}3416$ are constants chosen to satisfy boundary conditions. Using this form and relation (1.21) to fix B/A, equation (5.17) leads to $\lambda_S = 1713{\cdot}7$, $S = 1/1{\cdot}51$, and

$$AB = 1{\cdot}51(\lambda - 1713{\cdot}7). \quad (5.23)$$

The correct value of λ_S for the infinitesimal disturbance is 1707·8, indicating that (5.22) is an adequate approximation to the correct infinitesimal solution. The heat transport law (5.23) is in such good agreement with the observations reported in table 1 that one is tempted to extend this use of the power integrals beyond the range of steady convection. As discussed in the Introduction, a second instability occurs at $\lambda \doteqdot 18\,000$, leading to aperiodic convection and an abrupt change to a new linear law of heat transport. In all, five or six transitions appear in the data up to $\lambda \doteqdot 10^6$, where 'fully turbulent'

Transition	λ Theoretical	λ Experimental	S_i^{-1} Theoretical	AB/λ Theoretical	AB/λ Experimental
1	1 708	1 700 ± 80	1·51	0	0
2	17 600	18 000 ± 1,000	1·72	1·37	1·3 ± 0·1
3	61 000	55 000 ± 4 500	1·83	2·63	2·3 ± 0·1
4	170 000	170 000 ± 15 000	1·9−	4·29	3·4 ± 0·1
5	411 000	425 000 ± 20 000	1·9+	6·87	4·5 ± 0·1
6	855 000	860 000 ± 30 000	2·0	8·52	5·7 ± 0·1

Table 1. Convective heat transport at the experimental transitions computed from equation (5.24).

conditions seem finally to be achieved. If, as suggested by the success of the power integral, self-distortion of a disturbance is not very important, then perhaps the interaction between several different disturbances plays a small role in the determination of heat transport. To test this possibility we can determine the total heat transport which would result from the independent growth of first, second and higher Rayleigh-like modes of instability. That is,

$$AB = \sum_{1}^{\infty} (\lambda - \lambda_{S_i})/S_i, \quad (5.24)$$

where the S_i determine the slope of the growth curve for the mode i and λ_{S_i} is the value of λ for instability for that mode. Equation (5.24) has the form of the observed heat transport up to $\lambda \doteqdot 10^6$. The values of the S_i and λ_{S_i} have been computed in the same way as S and λ_S of (5.23) with the approximate functions of Pellew & Southwell for each instability. The results are tabulated in table 1 and compared with observations in water made by Malkus (1954a). In dimensional form AB/λ is the ratio of the convective heat transport to the heat transport in the absence of convection. For large values of λ, S_i^{-1} approaches its free surface value and has been set equal to this value for λ greater than 800000. Equation (5.24) agrees quite well with experiment at the end of the steady convective range, but by $\lambda \doteqdot 55\,000$ at the end of the first aperiodic range there is an error of approximately 15%. This percentage error continues to increase with λ through all observable transitions. Equation (5.24) cannot be correct in the range of λ beyond observable transitions since it leads to an incorrect gross dependence of heat transport on λ (giving $AB/\lambda \sim \lambda^{1/4}$ instead of the observed variation as $\lambda^{1/3}$). However it is rather remarkable that (5.24) works as well as it does. This suggests to us the possibility that aperiodicity permits the several disturbances to be more or less independent which certainly could not be the case if they were steady superimposed motions. The quasi-independence due to the aperiodicity could permit a greater release of potential energy and hence lead to a more stable field of motion than any other. Such thoughts are of course only speculation and a formal theory for the onset of thermal turbulence has yet to be found.

6. Concluding remarks

To summarise the conclusions: the initial finite amplitude convection is determined primarily by the distortion of the mean temperature field, secondarily by the self-distortion of the disturbance; the number of formal steady-state solutions increases with increase of Rayleigh number beyond the critical value; however this degeneracy is removed if there is only one finite amplitude solution which has a greater mean-square temperature gradient than any other solution. More particular conclusions are that: square plan-forms are preferred to hexagonal plan-forms in ordinary fluids with symmetric boundary conditions; the initial stages of convection are markedly altered in a fluid of small Prandtl number; the effect of distortion of the disturbance on the thermal field prevents the mean gradient of the temperature from changing sign in the mid-regions of the fluid; the zero-average non-linear terms reduce the amplitude of the disturbance; the preferred horizontal wave-number increases with increasing Rayleigh number.

The approach to finite amplitude processes given above is applicable to those convection and shear flow studies with adequately solved stability problems. The method can determine those conditions which permit a steady finite amplitude disturbance to occur before the appearance of the infinitesimal disturbance. Hence it is a tool to extend the study by Meksyn & Stuart (1952) of the onset of disturbances in channel shear flow. The method is of particular value in the resolution of explicit or hidden degeneracies in a stability theory. Hidden degeneracies exist when there are velocity-dependent forces other than advection which play no role in the criterion for instability. This occurs for example in the study of geomagnetism. There one wishes to determine when an electrically conductive fluid heated from below will select a joint magneto-convective state of motion rather than a pure convective state.

However, beyond the initial finite amplitude effects the technique we have described here can become prohibitively tedious. In addition the ϵ-sequence will fail to describe the preferred field of motion after a second instability has occurred, though this sequence may still converge to a set of formal steady-state solutions. We continue to search for an adequate descriptive framework which can include the second instability and in particular can deal with aperiodicity.

This study bears a relation to two previous works on fully developed turbulence. A paper on turbulent convection (Malkus 1954b) and a paper on turbulent shear flow (Malkus 1956) were based on the assumption that the most stable field of flow would be that one which released the maximum amount of potential energy per unit time. The section of this paper on relative stability suggests that a maximum mean-square temperature gradient (equation (4.14)) is the more appropriate stability criterion. The only qualitative prediction of the earlier works which appears to be altered by this change is the structure of the boundary regions and this causes a reduction in the turbulent transports. These modifications will be discussed in a following paper.

It was our good fortune that Dr J. T. Stuart was visiting the United States during the preparation of this paper. We wish to acknowledge many fruitful discussions of our common problem with him.

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